MECHANICAL ENGINEERING AND SOLID MECHANICS SERIES

Lagrangian Mechanics

An Advanced Analytical Approach

Anh Le van and Rabah Bouzidi

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Lagrangian Mechanics

To our parents;
to Nicole and Younnik, to Mai;
to Yemma Nounou, Shehrazed, Rayane, Redwan, Elyas

Series Editor
Noël Challamel

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Preface

There are two distinct yet equivalent approaches to solving a problem in rigid body mechanics: the Newtonian approach, based on Newton's laws, and Lagrange's approach, based on the postulate called the principle of virtual powers, and which lead to Lagrangian or analytical mechanics.

Although both approaches yield equivalent results, they differ on a number of points both in terms of conception as well as formulation. In addition to the usual ingredients – velocity, acceleration, mass and forces – analytical mechanics involves a new concept that does not exist in Newtonian mechanics, which is given the enigmatic name "virtual velocity". It is based on this concept that virtual powers are defined. While Newton's laws state relationships between vector quantities (force and acceleration), the principle of virtual powers, is written in terms of virtual powers, which are scalar quantities. Analytical mechanics is also distinguished by the fact that parameterization plays a primordial role here: given the same mechanical problem, it is possible to choose different parameterizations and the resulting equations – and thus, the information they yield – differ based on the chosen parameterization. Another salient feature demonstrated in analytical mechanics is that once the parameterization is chosen, the kinematical behavior of the system, a *vector* description in essence, is condensed into a *scalar* function, called the parameterized kinetic energy.

While Newtonian mechanics brings into play physical concepts that are easy to apprehend, Lagrange theory appears more complicated because of the virtual velocity and the very statement of the principle of virtual powers. However, two technical advantages compensate for this conceptual difficulty:

(i) It is seen that the physicist's task is practically reduced to choosing the appropriate parameterization for the system under study. Once the parameterization is chosen, the Lagrange equations systematically lead to as many equations as there are unknowns (if the problem is well-posed). Of course, it is also possible to obtain a sufficient number of equations using the Newtonian approach, but there is no systematic way of doing so. One must first carry out an analysis of the applied forces and must often subdivide the mechanical system being studied in an adequate manner and then write the equations for the subsystems.

(ii) The operations carried out in analytical mechanics – especially the calculation of the parameterized kinetic energy and its derivatives – are purely algebraic and, therefore, *programmable*. This explains the success of analytical mechanics in the study of complex systems containing a large number of kinematic parameters, where it is more difficult to obtain the equations of motion using the Newtonian approach.

This book strives to explain the subtleties of analytical mechanics and to help the reader master the techniques to obtain Lagrange's equations in order to fully use the potential of this elegant and efficient formulation. It is meant for students doing their bachelors or masters degree

in Engineering, who are interested in a comprehensive study of analytical mechanics and its applications. It is also meant for those who teach mechanics, engineers and anyone else who wishes to review the fundamentals of this field. Although the content does not require any prior knowledge of mechanics, it is preferable for the reader to be familiar with the Newtonian approach.

The format adopted in this book

When writing this book, the authors laid out the objective of revisiting analytical mechanics and presenting it from a different angle both in form and style. This was done:

– by adopting a more axiomatic and formal framework than a conventional course,

– and by taking special efforts concerning notations to arrive at mathematical expressions that are both precise and concise.

By "axiomatic framework", we mean that all through this book, the chapters are constructed in a manner that is similar to a mathematical discussion, where the ingredients are presented in the following order:

1. the definitions to establish the vocabulary used,

2. the theorems, where results are proven and where we specify the hypotheses, clearly indicating the conditions of applicability for this or that result,

3. and finally, examples to illustrate the nuances of the theory and the mechanisms of the calculation.

While the theory is constructed in a deductive manner and forms a monolithic block, each theorem is written in a self-contained and condensed form – that is, hypotheses followed by results – in order to make it "ready to use".

Synopsis

This book contains 11 chapters and two appendices:

Chapter 1 reviews the basic ingredients of kinematics: time, space and the observer (or reference frame). We present here the key concept of the derivative of a vector with respect a reference frame and introduce the concept of a "common reference frame", which is used to connect or relate two different reference frames.

Chapter 2 focuses on an important operation in analytical mechanics, namely the parameterization of the mechanical system being studied. This operation consists of choosing a certain number of primitive parameters of the system, expressing all existing constraints in terms of these parameters and, finally, classifying the constraint equations into two categories, called "primitive" and "complementary" equations. This task, incumbent on the physicist, is specific to analytical mechanics and has no equivalent in Newtonian mechanics. It is important because, as we will see, the Lagrange equations that are obtained (and, consequently, the information that may be extracted from them) are essentially dependent on the choice of parameterization.

The parameterization of the system leads to the definition of the parameterized velocities and the parameterized kinetic energy, the concept on which the Lagrange kinematic formula is based.

Chapter 3 reviews the conventional concept of efforts that includes forces and torques. These can be classified as either internal efforts and external efforts, or given efforts and constraint efforts. The virtual powers of efforts are calculated in Chapter 5 depending on how the efforts are categorized.

Chapter 4, dedicated to virtual kinematics, introduces new kinematic quantities that are the counterparts of those introduced in Chapter 1 and are labeled "virtual": the virtual derivative of

a vector with respect to a reference frame, the virtual velocity of a particle, the virtual velocity fields in a rigid body or a system of rigid bodies, and, finally, the virtual angular velocity of a rigid body. This chapter provides formulae to calculate these quantities and, notably, the formulae for the composition of virtual velocities.

Chapter 5 deals with virtual power, which is, *grosso modo*, the product of an effort, seen in Chapter 3, and a virtual velocity, seen in Chapter 4. The presentation closely follows the conventional presentation of real powers in Newtonian mechanics and we obtain several results that are analogous to those obtained for real powers. Two results are, nevertheless, specific to analytical mechanics: the virtual power expressed as a linear form and the power of the quantities of acceleration.

Chapter 6 shows how to exploit the principle of virtual powers using the results obtained in the previous chapters in order to arrive at the final product of analytical mechanics, namely the famous Lagrange equations. In this chapter, we also see several examples which illustrate how important the choice of parameterization is and what consequences it has on the obtained results. This chapter concludes with the statement of Lagrange equations in a non-Galilean reference frame.

Chapters 7 and 8 are concerned with perfect joints. The chief advantage of these joints is that the generalized forces present in the right-hand side of the Lagrange equations are then zero or may be easily calculated using Lagrange multipliers. The concept of the perfect joint also exists in Newtonian mechanics, but they are defined there in a simpler manner, with more obvious consequences. In analytical mechanics, the definition of a perfect joint is less natural inasmuch as it involves the parameterization and the virtual velocities that are compatible with the complementary constraint equations. It is, therefore, important to verify that the perfect character of a joint is intrinsic, i.e. it does not depend on the chosen parameterization. For this reason, a large section is dedicated to the invariance of virtual velocity fields with respect to the parameterization.

Chapter 9 is dedicated to the first integrals, which offer the advantage of yielding first-order differential equations that are easier to solve. The first integral called "Painlevé's first integral" has no equivalent in Newtonian mechanics and presents the unique feature of being able to exist for systems that receive energy from the exterior. The energy integral resembles that of Newtonian mechanics. However, the conditions for application are slightly different.

Chapter 10 shows how the Lagrange equations are simplified in the particular case of equilibrium. The chapter also contains a brief discussion on the question of the stability of an equilibrium position.

Chapter 11 contains several examples to revise all the concepts presented in the book.

The book concludes with two appendices. Appendix1 provides a few basic concepts related to second-order tensors, which are necessary for studying kinematics.

Appendix2 complements Chapter 7 and establishes the necessary and sufficient conditions of perfectness for joints that are usually encountered in mechanics.

By introducing the concept of virtual quantities, Lagrangian mechanics is more abstract than Newtonian mechanics. Nonetheless, it proves to be more fertile in that it extends beyond the mechanics of rigid bodies in order to lead to more elegant and systematic formulations in the field of mechanics of deformable media, to say nothing of physics in general. This is why analytical mechanics is one of the fundamental subjects taught in mechanics. The authors of this book hope to offer the reader a comprehensive view of and perfect mastery over this beautiful formulation.

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taken from the team's tutorial archive, are the result of long reflection and discussions between several colleagues, some of whom are no longer with us.

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ANH LE VAN AND RABAH BOUZIDI
Nantes, France, March 2019

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Kinematics

Mechanics, the science that studies the motion of bodies, generally comprises three parts: (i) kinematics, where motions are described regardless of the causes that provoked the motions; (ii) kinetics, where we combine kinematics with the concept of mass; and (iii) dynamics, where we add the concept of the forces acting on the body.

This chapter will focus on kinematics and will review the essential concepts of classical mechanics: the observer or reference frame, time, space, as well as velocity and acceleration. Chapter 2 will focus on the parameterization, the parameterized kinematics as well as the parameterized kinetic energy, concepts that are essential for analytical mechanics. Forces will be studied in Chapter 3.

Apart from the above-mentioned real quantities, analytical mechanics also brings in virtual quantities (virtual velocity, virtual power), which will be discussed in Chapters 4 and 5.

1.1. Observer – Reference frame

We admit the existence of *observers*, real or fictional, located in areas that may or may not be accessible to humans. An observer is denoted by $\mathcal{O}_i$, the integer index i making it possible to distinguish from the different observers.

The observer $\mathcal{O}_i$ needs an instrument called a *clock* to note the start, end or duration of an event (the notions of event and time will be seen hereafter).

They also need an *observation instrument* placed on a *support* called a *reference solid*, to observe mechanical systems and their positions in physical space and at each instant (Figure 1.1; the notions of mechanical system and position will be seen in section 1.3).

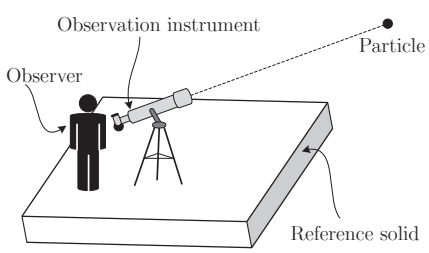

Figure 1.1. *Reference solid and observation instrument*

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Definition. A *reference frame* is an observer equipped with a clock and an observation instrument placed on a reference solid. The reference frame associated with the observer $\mathcal{O}_i$ will be denoted as R_i.

Thus, the term "reference frame" is almost synonymous with "observer", while having a slightly more precise sense. Furthermore, the term "observer" makes one think of the presence of a human in the study, whereas "reference frame" is more impersonal. For notational convenience, the notation R_i is preferred to $\mathcal{O}_i$ in mathematical relationships.

1.2. Time

Each observer possesses the following concepts with respect to time: (1) the consciousness *in the moment* when a given instantaneous event takes place; (2) the perception of the *chronological order*, of *anteriority* or *posteriority*, and, consequently, of the *simultaneity* of two instantaneous events; (3) and finally, the perception of the *duration* of an event.

In order to carry out calculations, the observer must transcribe the data from the clock into a mathematical set. It is decided that this set is a one-dimensional affine space called (mathematical) "*time*", which is the set $\mathbb{R}$ of all real numbers equipped with (1) the partial order $\leq$ in order to account for the chronological order (anteriority or posteriority) and (3) a structure of vector space or affine space, which makes it possible to represent the duration by a scalar.

1.2.1. *Date postulate*

The following postulate shows how an observer passes from the set of instantaneous events to the time set:

$$\text{Set of instantaneous events} \quad \rightsquigarrow \quad \text{Time } \mathbb{R}$$

Date postulate. [1.1]

An observer $\mathcal{O}_i$ possesses a clock which enables them to match each instantaneous event with one and only one scalar, $t^{(i)}$, called the *instant* of an event observed by the observer $\mathcal{O}_i$ (or with respect to the reference frame R_i).

In short, the observer $\mathcal{O}_i$ is able to mark an instantaneous event with an instant.

The upper index (i) in $t^{(i)}$ reminds us that the involved instants are observed with respect to the reference frame R_i.

1.2.2. *Date change postulate*

In a mechanical problem with multiple observers, each observer $\mathcal{O}_i$ has his own clock that allows him to mark an instantaneous event by an instant $t^{(i)}$ in $\mathbb{R}$. The question which then arises is that of communication between these observers and more precisely, that of the correspondence between their different observed instants. The date change postulate makes it possible to establish a relation between simultaneously observed instants with respect to different reference frames:

Date change postulate. Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two arbitrary observers. According to the date postulate [1.1], it is possible to know the instants $t^{(1)}$ and $t^{(2)}$ marked by $\mathcal{O}_1$ and $\mathcal{O}_2$, respectively, and corresponding to the *same* instantaneous event.

When we consider different instantaneous events, we obtain different corresponding couples $(t^{(1)}, t^{(2)})$. It is assumed that there exists a *continuous and strictly increasing* mapping, χ_{21}: $\mathbb{R} \to \mathbb{R}$, that gives the date $t^{(2)}$ as a function of the date $t^{(1)}$:

$$\boxed{t^{(2)} = \chi_{21}(t^{(1)})} \qquad [1.2]$$

Using the mapping χ_{21}, we know the correspondence between the instants noted by the two observers. If we know the instant $t^{(1)}$, noted by the observer $\mathcal{O}_1$, we may deduce the instant $t^{(2)}$ noted *simultaneously* by the observer $\mathcal{O}_2$, by making $t^{(2)} = \chi_{21}(t^{(1)})$, and vice versa. The term "simultaneously" signifies that the two dates correspond to the *same* instantaneous event.

Since the application χ_{21} is continuous and strictly increasing, it is a homeomorphism (i.e. bicontinuous bijection) from $\mathbb{R}$ to $\mathbb{R}$. The most natural choice is to take χ_{21} equal to an affine function or, even more simply, to take it equal to the identity function. In other words, it is assumed that all observers mark the same instantaneous event with the same scalar:

$$\boxed{t^{(1)} = t^{(2)} = \cdots = t} \qquad [1.3]$$

An important consequence of this choice is

$$\boxed{\frac{d}{dt^{(1)}} = \frac{d}{dt^{(2)}} = \ldots = \frac{d}{dt}} \qquad [1.4]$$

This is why we will only encounter a single derivative with respect to time $\dfrac{d}{dt}$ from now on.

Eventually, a single clock is enough for all observers and this is what we will assume in the sequel.

1.3. Space

1.3.1. *Physical space*

The *physical space* (or the material world) is composed of vacuum and *matter*. It is common to all observers (who are embedded in the same physical space). It is intrinsic in the sense that it exists even in the absence of observers.

1.3.1.1. *Mechanical system*

A *mechanical system* is loosely defined as an invariant collection of matter. This definition is an intuitive one, but is not rigorous as the word *matter* has not been defined.

As with physical space, the concept of a mechanical system is intrinsic and has been defined independently of the observer.

1.3.1.2. *Particle*

It is assumed that with the help of an observation instrument, the observer $\mathcal{O}_i$ is able to distinguish, within the physical space, mechanical systems (or mechanical subsystems) that they consider to be small. This kind of system is called *a particle* (or *material point*) for the observer $\mathcal{O}_i$.

This definition is not precise because we do not know how the observer may evaluate that a system is small. The particle is a *model*, that is a choice made by the physicist of how to represent the system under consideration, with respect to the nature of the problem being studied and the objective that is fixed, and it is the simplest model in mechanics.

With the concept of the particle having been defined for a given observer, it can be assumed that the concept of particle is, in fact, invariant for all observers, i.e. *in a given problem, a system perceived to be a particle will be a particle for any other observer.*

• We consider that any mechanical system is a *union, which may or may not be finite, of particles*. As the family of particles in question is always the same, it is assured that we have an invariant collection of matter.

In this book, a mechanical system is made up of a finite number of rigid bodies (of which some may be reduced to particles).

1.3.2. *Mathematical space*

In order to carry out calculations, the observer must transcribe the results of their observation over time into mathematical data. To prepare for this task, we introduce a new mathematical structure, apart from time, which is defined as follows:

Definition. The *mathematical space*, denoted by $\mathcal{E}$, is a three-dimensional real affine space, given beforehand. [1.5]

As will be seen in the position postulate [1.10], all observers use the mathematical space $\mathcal{E}$ in order to enter their observations from the physical space. The following terms associated with the space $\mathcal{E}$ are well known in mathematics.

Definition. [1.6]

- An element $A \in \mathcal{E}$ is called *a point*.
- *A bi-point* or *a vector* is the difference $B - A \equiv \overrightarrow{AB}$ between two points $A, B \in \mathcal{E}$.
- *The vector space* is the vector space E associated with $\mathcal{E}$.
- *A coordinate system* of $\mathcal{E}$ is defined by a point $O \in \mathcal{E}$ and a basis $e = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ of E that is made up of three independent vectors. We denote it by $(O; e)$ or $(O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$. The point O, which is arbitrarily chosen in $\mathcal{E}$, is called the *origin* of $\mathcal{E}$ or *of the coordinate system*.
- If $\overrightarrow{OA} = \sum_{i=1}^{3} x_i \vec{e}_i$, then (x_1, x_2, x_3) are called *the components* of the vector $\overrightarrow{OA}$ in the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, or *the coordinates* of the point A in the coordinate system $(O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$.

By choosing, beforehand, an origin O in $\mathcal{E}$ and a basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ in E, in other words, by choosing, beforehand, a coordinate system $(O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in $\mathcal{E}$, we have the following bijections:

$$\begin{array}{ccccc} \mathcal{E} & \to & E & \to & \mathbb{R}^3 \\ A & \mapsto & \overrightarrow{OA} = \sum_{i=1}^{3} x_i \vec{e}_i & \mapsto & (x_1, x_2, x_3) \end{array} \qquad [1.7]$$

1.3.3. *Position postulate*

Definition. [1.8]

A physical coordinate system for the observer $\mathcal{O}_i$ (or reference frame R_i) is the quadruplet $\boxed{(o_i; a_i, b_i, c_i)}$ made up of four *non-coplanar* particles (real or fictitious, i.e., materialized or non-materialized) taken *in* the reference solid (Figure 1.2(a)).

Keep in mind that the order of the particles, o_i, a_i, b_i, c_i, is important: the first particle o_i is, by definition, the *origin of the physical coordinate system*, the particles a_i, b_i, c_i will correspond, respectively, to the three vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ of the basis E, as will be seen in [1.14b].

The physical coordinate system $(o_i; a_i, b_i, c_i)$ should be distinguished from the mathematical coordinate system $(O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in $\mathcal{E}$.

Any couple formed by two particles will be called a *physical segment* and $[pq]$ will denote the segment formed by two particles p, q. It is assumed that by means of their observation instrument,

the observer is able to measure the distance between any two particles p, q and this distance is called the *physical length of the segment* $[pq]$, and also that they are able to measure the angle between any two segments. Two segments are said to be *physically orthogonal* when the angle between them is $90°$ (the right angle may be detected by using, for example, a plumb line and the water level at the same location or by using a protractor).

Definition. *The physical coordinate system* $(o_i; a_i, b_i, c_i)$ is said to be *physically orthonormal* if the segments $[o_i a_i]$, $[o_i b_i]$, $[o_i c_i]$ are of unit length and mutually physically orthogonal. [1.9]

The following postulate stipulates how an observer, at every instant, carries out the passage from the physical space to the mathematical space:

$$\text{Physical space} \quad \rightsquigarrow \quad \text{Mathematical space } \mathcal{E}$$

Position postulate in R_i [1.10]

(a) The observer $\mathcal{O}_i$ possesses an observation instrument placed on the reference solid, by means of which they are able, at each instant, to locate a particle in the physical space with a point in the space $\mathcal{E}$. More precisely, at each fixed instant t, they are able to establish a correspondence between each particle p in the physical space and a single point $P^{(i)}$ in mathematical space $\mathcal{E}$. This point is denoted by $\boxed{P^{(i)} \equiv pos_{R_i}(p,t)}$ and is read as *the position of the particle p with respect to R_i (or in R_i) at instant t*. The upper index (i) reminds us that this involves results coming from the observer R_i.

The statement is then written as

$$\forall t, \forall \text{ particle } p, \exists! P^{(i)} \in \mathcal{E}, P^{(i)} \equiv pos_{R_i}(p,t) \qquad [1.11]$$

(b) Conversely, given a fixed instant t, any point in the mathematical space $\mathcal{E}$ is the position in R_i of at least one particle, real or fictitious (that is, materialized or not):

$$\forall t \in \mathcal{T}, \forall A \in \mathcal{E}, \exists \text{ particle } p, A = pos_{R_i}(p,t) \qquad [1.12]$$

(c) **Convention on the physical coordinate system of R_i** [1.13]

The following clauses concern the physical coordinate system and include the hypotheses that are part of the postulate as well as the conventions that aim to simplify the exposition.

Let $(o_i; a_i, b_i, c_i)$ be a physical coordinate system for the reference frame R_i (see definition [1.8]), and let us denote the respective positions of the particles o_i, a_i, b_i and c_i in R_i at instant t by $O_i^{(i)}$, $A_i^{(i)}$, $B_i^{(i)}$ and $C_i^{(i)}$:

$$\boxed{O_i^{(i)} \equiv pos_{R_i}(o_i,t) \quad A_i^{(i)} \equiv pos_{R_i}(a_i,t) \quad B_i^{(i)} \equiv pos_{R_i}(b_i,t) \quad C_i^{(i)} \equiv pos_{R_i}(c_i,t)}$$

- The observer $\mathcal{O}_i$ chooses their physical coordinate system $(o_i; a_i, b_i, c_i)$, physically orthonormal in the sense of definition [1.9].
- The positions in R_i of the four particles o_i, a_i, b_i, c_i are points in $\mathcal{E}$ that are *fixed* over time. To simplify, we will make them equal to the fixed points that make up the mathematical space $\mathcal{E}$ as follows:
  - $\forall t$, let us take the position of the origin o_i, in R_i, of the physical coordinate system equal to the origin O in $\mathcal{E}$ (Figure 1.2):

$$\forall t, \boxed{O_i^{(i)} = O} \qquad [1.14]$$

○ It is assumed that it is possible to take

$$\forall t, \boxed{\begin{array}{l} A_i^{(i)} = O + \vec{e}_1 \\ B_i^{(i)} = O + \vec{e}_2 \\ C_i^{(i)} = O + \vec{e}_3 \end{array}} \Leftrightarrow \boxed{\begin{array}{l} \overrightarrow{OA_i^{(i)}} = \vec{e}_1 \\ \overrightarrow{OB_i^{(i)}} = \vec{e}_2 \\ \overrightarrow{OC_i^{(i)}} = \vec{e}_3 \end{array}}, \qquad [1.14b]$$

where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are the vectors of the basis of E.

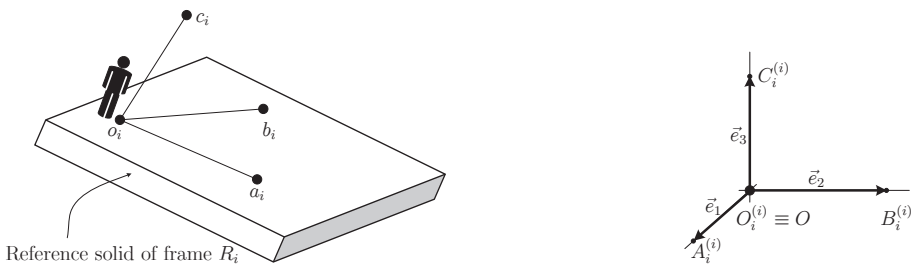


Figure 1.2. *The position in R_i of the physical coordinate system for R_i*

The definition of the position of a body or a mechanical system is based on that of a particle:

Definition. [1.15]

The position of a body S, in the reference frame R_i and at an instant t, denoted $\boxed{pos_{R_i}(S,t)}$, is, by definition, the set of positions $P^{(i)}$, at t, of all the particles in S. This is a subset of $\mathcal{E}$.

If S is a finite union of particles, $pos_{R_i}(S,t)$ is a discrete subset of $\mathcal{E}$ made up of a finite number of points. If not, $pos_{R_i}(S,t)$ may be a volume, a surface or a curve in $\mathcal{E}$ and we then say that the body S is a *volumetric, surface or line body*.

Definition. [1.16]

Let us consider a mechanical system $\mathcal{S}$, made up of a finite number of bodies (some of which may be reduced to particles). The position, in a reference frame R_i and at an instant t, denoted by $\boxed{pos_{R_i}(\mathcal{S},t)}$, is, by definition, the set of positions $P^{(i)}$, at t, of all particles in $\mathcal{S}$, in other words, the set of positions in R_i at t of all the constituent bodies. This is a subset of $\mathcal{E}$.

If $\mathcal{S}$ is a finite union of particles, $pos_{R_0}(\mathcal{S},t)$ is a discrete subset of $\mathcal{E}$. Otherwise, $pos_{R_0}(\mathcal{S},t)$ may be a union of volumes, surfaces or curves in $\mathcal{E}$ and we then say that the mechanical system $\mathcal{S}$ is a *volumetric, surface or line system*.

1.3.4. *Typical operations on the mathematical space $\mathcal{E}$*

We will review here the mathematical operations that are typically carried out on the mathematical space $\mathcal{E}$, together with the interpretations and the corresponding physical operations.

• **Endowing E with a structure of Euclidean space**

Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be a basis of E. It can be easily verified that the following bilinear mapping defined on E is a scalar product (that is, a bilinear form, that is symmetric and positive definite):

$$\begin{array}{ccc} E \times E & \rightarrow & \mathbb{R} \\ (\vec{x} = \sum_{i=1}^{3} x_i \vec{e}_i, \vec{y} = \sum_{i=1}^{3} y_i \vec{e}_i) & \mapsto & \vec{x}.\vec{y} \equiv \sum_{i=1}^{3} x_i y_i \end{array} \qquad [1.17]$$

The space E equipped with this scalar product is a Euclidean space. The definition [1.17] implies that

$$\forall i, j = 1, 2, 3, \quad \vec{e}_i.\vec{e}_j = \delta_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \qquad (\delta_{ij} \text{ is the Kronecker symbol})$$

that is, the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is *orthonormal*.

We can now understand why one had better choose a physical coordinate system that is physically orthonormal, as was done in convention [1.13]. The coordinate system of $\mathcal{E}$ is the image of the physically orthonormal physical coordinate system and this is *consistent* with the fact that the basis in E is orthonormal.

The scalar product [1.17] makes it possible to define the following norm in E denoted by $\|.\|$:

$$\begin{array}{ccc} E & \rightarrow & \mathbb{R} \\ \vec{x} = \sum_{i=1}^{3} x_i \vec{e}_i & \mapsto & ||\vec{x}|| \equiv \sqrt{\vec{x}.\vec{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2} \end{array} \qquad [1.18]$$

The scalar $||\vec{x}||$ is called *the norm* or *the magnitude of vector* $\vec{x}$.

• **Orienting E and $\mathcal{E}$**

The observer $\mathcal{O}_i$ names the four particles o_i, a_i, b_i, c_i of their physical coordinate system according to the right-hand rule, i.e. in such a way that when they are placed along $o_i a_i$ (their feet on o_i and their head at a_i) and when they are looking toward b_i, they have c_i on their left. The observer $\mathcal{O}_i$ orients E and $\mathcal{E}$ by deciding that $(\overrightarrow{O_iA_i}, \overrightarrow{O_iB_i}, \overrightarrow{O_iC_i})$ is a right-handed orthonormal basis. Such a right-handed orthonormal basis is represented in Figure 1.2.

Throughout the sequel, we will work only with right-handed orthonormal bases.

1.3.5. *Position change postulate*

The difficulty when several observers come into play is establishing the relationship between their different observation results. Indeed, as the observers choose their reference solids independently of each other, it turns out that even if they observe the same physical space, *there is,* a priori*, no relationship between the positions observed by various observers*.

To illustrate this fact, let us consider two observers $\mathcal{O}_1$ and $\mathcal{O}_2$, or two reference frames R_1 and R_2. *Let us fix an instant* t (this has a sense according to hypothesis [1.3]) and let us consider a particle p in the physical space (Figure 1.3(a)).

The observers learn the same mathematics and they use the same mathematical space $\mathcal{E}$ where they write down the results of their observation from the physical space. According to the position postulate [1.11], the observer $\mathcal{O}_1$ may mark *the position of* p *in* R_1*, at* t, as $P^{(1)} = pos_{R_1}(p, t)$, this is a point in $\mathcal{E}$ (Figure 1.3(b)). The observer $\mathcal{O}_2$ may, in turn, mark *the position of* p *in* R_2*, at* t, as $P^{(2)} = pos_{R_2}(p, t)$, that is, *a priori*, another point in $\mathcal{E}$ (Figure 1.3(b)).

There arises the following question: what is the relationship between the two points $P^{(1)}$ and $P^{(2)}$ in $\mathcal{E}$? In other words, what is the relationship that gives the position of a particle in R_2 as a function of the position of the same particle in R_1 at the same instant?

The following postulate, called the position change postulate, makes it possible to connect the observed positions in different reference frames for the same particle at a given instant.

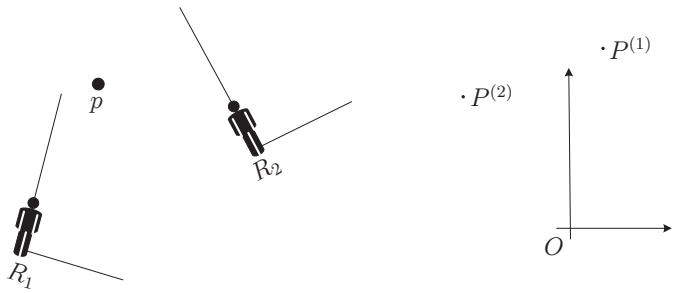


Figure 1.3. *Positions of a particle, observed at the same instant by two different observers*

Position change postulate. $\forall$ reference frames R_1 and R_2, $\forall$ instant t, $\exists$ *a positive isometry* $Q_{21}(t)$: $\mathcal{E} \to \mathcal{E}$ such that $\forall$ particle p, located, respectively, at $P^{(1)} \equiv pos_{R_1}(p,t), P^{(2)} \equiv pos_{R_2}(p,t)$ in R_1 and R_2, we have

$$\boxed{P^{(2)} = Q_{21}(t)P^{(1)}} \qquad [1.19]$$

where the right-hand side is the image of point $P^{(1)}$ under $Q_{21}(t)$.

In other words, at every instant t, the positions of the same particle, observed in the two reference frames R_1 and R_2, are connected by a positive isometry denoted by $Q_{21}(t)$.

The bijection $Q_{21}(t)$ constitutes a "dictionary", with the help of which the two observers are able to establish correspondence between their observed positions. This dictionary *varies with time*.

REMARK. According to [1.19], if the position of a particle p is *fixed* in R_1, that is, if the point $P^{(1)} = pos_{R_1}(p,t)$ is a fixed point in $\mathcal{E}$, then the position ($P^{(2)}$) of this particle in R_2 will be, *a priori*, a point that varies with time and vice versa. ■

We will need the following terminology:

Definition. *A biposition* is the difference between two positions of two particles (just as a bipoint is the different between two points). [1.20]

Using the position change postulate [1.19], we can establish correspondence between the bipositions observed in two different reference frames. In order to do this, let us use the following mathematical result, which is well known for a point isometry:

Theorem and definition. $Q_{21}(t)$ is a point isometry, $\Leftrightarrow$ $Q_{21}(t)$ is *affine* and its linear part denoted by $\bar{\bar{Q}}_{21}(t) : E \to E$ is a *vector* isometry.

We then have $\forall t, \forall A, B \in \mathcal{E}$

$$\boxed{Q_{21}B - Q_{21}A = \bar{\bar{Q}}_{21}.\overrightarrow{AB}} \qquad [1.21]$$

where the right-hand side is the product of $\bar{\bar{Q}}_{21}$ and vector $\overrightarrow{AB}$.

$\bar{\bar{Q}}_{21}(t)$ is, by definition, the *rotation tensor of* R_1 *with respect to* R_2 *at instant* t or *the reference frame change tensor* (see Appendix 1 for a brief review of tensor algebra).

From this, we have the following result, which is the vector version of the point relationship [1.19] :

Theorem of biposition change. At a given instant t,
let $P^{(1)}, P^{(2)}$ be the positions of a particle, observed, respectively, in R_1, R_2, and
let $Q^{(1)}, Q^{(2)}$ be the positions of another particle, observed, respectively, in R_1, R_2.
We have

$$\boxed{\overrightarrow{P^{(2)}Q^{(2)}} = \bar{\bar{Q}}_{21}(t).\overrightarrow{P^{(1)}Q^{(1)}}} \qquad [1.22]$$

PROOF. It follows immediately from the position change postulate [1.19] and from theorem [1.21] that

$$\overrightarrow{P^{(2)}Q^{(2)}} = Q^{(2)} - P^{(2)} = Q_{21}Q^{(1)} - Q_{21}P^{(1)} = \bar{\bar{Q}}_{21}(t).\overrightarrow{P^{(1)}Q^{(1)}} \qquad \blacksquare$$

The following theorem brings together three very useful properties that isometries possess:

Theorem. $\forall$ reference frames $R_1, R_2, R_3, \forall t$,

$$\begin{array}{l} \boxed{Q_{31} = Q_{32}.Q_{21}} \\ \boxed{Q_{11} = I} \\ \boxed{Q_{21}.Q_{12} = I} \Leftrightarrow \boxed{Q_{12} = Q_{21}^{-1} = Q_{21}^{T}} \end{array} \qquad [1.23]$$

where I is the identity function and the symbol T denotes the transpose.

These equalities are also valid when we replace the point isometries Q_{ij} with the rotation tensors $\bar{\bar{Q}}_{ij}$.

1.3.6. *The common reference frame* R_0

A single event can be observed in different reference frames $(R_1, R_2, R_3, \ldots)$ at different instants $(t^{(1)}, t^{(2)}, t^{(3)}, \ldots)$, connected by the date change postulate [1.2]. There is no difficulty even if there are many of these instants as we have assumed, in [1.3], that they are equal: $t^{(1)} = t^{(2)} = t^{(3)} = \cdots$. This makes it possible to use the same symbol t to denote all of them.

With regard to the space, however, the situation becomes a little more complex. At a given instant t, a particle p is observed at a multitude of positions $P^{(1)}, P^{(2)}, P^{(3)}, \ldots$, which are related through the position change postulate [1.19]. All these points $P^{(1)}, P^{(2)}, P^{(3)}, \ldots$ are elements of the (mathematical) space $\mathcal{E}$, and they are, *a priori*, distinct and cannot be confused (contrary to what is done with instants).

Since the notations multiply rapidly with an increase in the number of the involved particles and reference frames, it is practically impossible to represent, in space $\mathcal{E}$, all the observation results in the different reference frames. For example, if there are two particles p, q and three reference frames R_1, R_2, R_3, we have, at a given instant, six positions represented in Figure 1.4. Thus, the figures quickly become indecipherable and too tedious to make.

Fortunately, it can be seen that there is no need to represent the observations in all the existing reference frames. For example, the observations in R_1, R_2 give rise to two different sets of points; however, they are, in fact, identical within an isometry, which is the isometry Q_{21} introduced in [1.19]. As a result, regardless of the number of reference frames in play, we can content ourselves with using the observation results from *one single reference frame* that is arbitrarily chosen from all the reference frames.

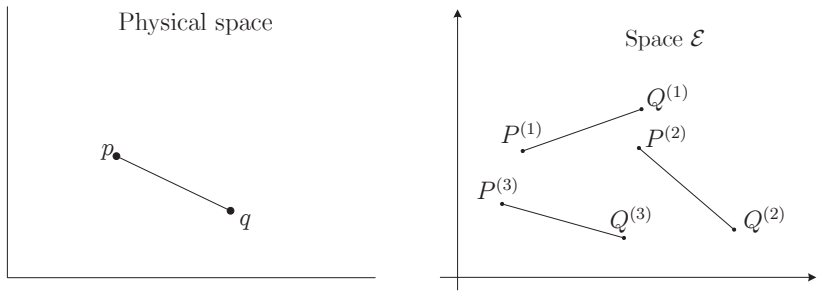


Figure 1.4. *Multiple positions in the presence of multiple reference frames*

Definition and notational convention. [1.24]

We arbitrarily choose one among the existing reference frames. This is called the *common reference frame* and denoted by $\boxed{R_0}$. *As it is enough to represent the observed positions in a single reference frame, we will choose to do this in the common reference frame,* R_0.

We agree to give R_0 a special status by simplifying the notation in R_0 as follows: the position $P^{(0)} = pos_{R_0}(p,t)$ of a particle p at an instant t, observed in the common reference frame R_0, will be denoted by P, *without the index* (0) *for the reference frame*:

$$\boxed{P = pos_{R_0}(p,t)}$$

(while the position $P^{(i)}$ in another reference frame R_i must include the index (i)).

Simply put, *we will represent only positions in* R_0 *and these positions will not include the index* (0) *for the reference frame*.

In the example in Figure 1.4, by choosing $R_0 = R_1$ we obtain Figure 1.5, which is more legible.

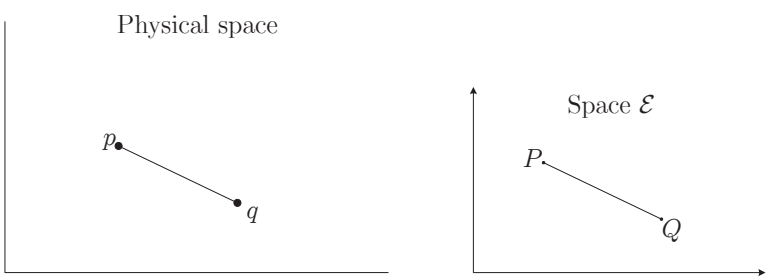


Figure 1.5. *Representation of the problem in Figure 1.4 using the common reference frame* R_0

In a mechanical problem that involves only a single reference frame, the problem of choosing a common reference frame does not arise at all. The reference frame R_0 is the only available one. On the other hand, if several reference frames come into play, we may wonder which one is to be chosen as the common reference frame R_0. While the choice is arbitrary in principle,

in practice the nature of the problem often causes a particular reference frame to come up and naturally impose itself as the most convenient reference frame for the role of R_0: this is the reference frame – i.e. the observer – with which the physicist identifies from the start and this is the reference frame that will be chosen as the common reference frame R_0.

The reference frame R_0 is common in the sense that it is shared by everyone. It may also be referred to as the *script reference frame* in the sense that, in general, the positions which appear in the mathematical relationships are the positions in R_0.

• By applying relationship [1.22], with $R_1 = R_i, R_2 = R_0$ and by taking into consideration the notation convention [1.24], we obtain:

$$\overrightarrow{PQ} = \bar{\bar{Q}}_{0i}(t).\overrightarrow{P^{(i)}Q^{(i)}} \qquad [1.25]$$

By making $Q^{(i)} = O_i^{(i)} \equiv pos_{R_i}(o_i, t)$ in this relationship, and by taking into account [1.14], $O_i^{(i)} = O$, we obtain

$$\overrightarrow{O_iP} = \bar{\bar{Q}}_{0i}(t).\overrightarrow{OP^{(i)}} \qquad [1.26]$$

• Using the notation convention [1.24], the convention on the physical coordinate system [1.13] is written in R_0 as follows:

Convention on the physical coordinate system of R_0. [1.27]

Let $(o_0; a_0, b_0, c_0)$ be a physical coordinate system for the reference frame R_0. Let O_0, A_0, B_0 and C_0 denote the respective positions of the particles o_0, a_0, b_0 and c_0, in R_0 and at the current instant t:

$$\boxed{O_0 \equiv pos_{R_0}(o_0, t) \qquad A_0 \equiv pos_{R_0}(a_0, t) \qquad B_0 \equiv pos_{R_0}(b_0, t) \qquad C_0 \equiv pos_{R_0}(c_0, t)}$$

- The observer R_0 chooses their physical coordinate system $(o_0; a_0, b_0, c_0)$, physically orthonormal as per definition [1.9].
- The positions of the four particles o_0, a_0, b_0, c_0 in R_0 are the points in $\mathcal{E}$ that are *fixed* over time. For simplicity, they will be equaled to the fixed points forming the coordinate system of $\mathcal{E}$ as follows:
  - $\forall t$, it is agreed that the position in R_0 of the origin o_0 of the physical coordinate system is equal to origin O of $\mathcal{E}$:

$$\forall t, \boxed{O_0 = O} \qquad [1.28]$$

  - It is assumed that it is possible to take

$$\forall t, \boxed{\begin{array}{l} A_0 = O + \vec{e}_1 \\ B_0 = O + \vec{e}_2 \\ C_0 = O + \vec{e}_3 \end{array}} \Leftrightarrow \boxed{\begin{array}{l} \overrightarrow{OA_0} = \vec{e}_1 \\ \overrightarrow{OB_0} = \vec{e}_2 \\ \overrightarrow{OC_0} = \vec{e}_3 \end{array}}, \qquad [1.28b]$$

where $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ are the vectors of the basis of E.

EXAMPLE. Let us consider the *bidimensional* problem of a disc S_2, of unit radius, rolling on a plane support S_0, as shown in Figure 1.6(a). The common reference frame R_0 is chosen as the frame whose reference solid is the support S_0 (this seems the most natural thing to do here). The physical coordinate system of R_0 is defined by three particles $[o_0; a_0, b_0]$ as shown in Figure 1.6(a).

We also define the reference frame R_2 whose reference solid is the disc S_2. The physical coordinate system in R_2 is defined by three particles $[o_2; a_2, b_2]$, and the particle o_2 is located at the center of the disc S_2 and the particles a_2, b_2 are on the edge of the disc.

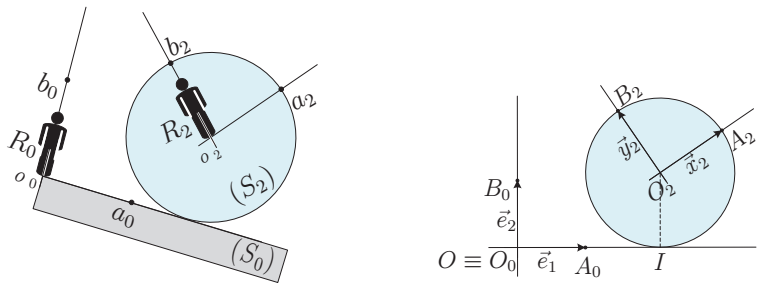


Figure 1.6. *(a) Disc rolling on a plane support; (b) positions of the systems in R_0*

According to convention [1.13], the physical coordinate systems are chosen such that they are physically orthonormal in the sense of definition [1.9]. The (mathematical) space $\mathcal{E}$, in which the positions in different reference frames are written, as well as the associated vector space E are two dimensional. The (mathematical) coordinate system of $\mathcal{E}$ is denoted by $(O; \vec{e}_1, \vec{e}_2)$.

According to the notation convention [1.24], we will represent the positions of the mechanical systems in R_0. By applying [1.24] and then $[1.14]_{a-b}$, we find that the positions in R_0 of particles o_0, a_0 are:

$$\begin{aligned} O_0 &= O_0^{(0)} = O \\ A_0 &= A_0^{(0)} = O + \vec{e}_1 \end{aligned}$$

Similarly, we find that the position in R_0 of particle b_0 is $B_0 = O + \vec{e}_2$. This problem is represented in Figure 1.6(b). ■

1.3.7. *Coordinate system of a reference frame*

Let us introduce another terminology that is commonly used in mechanics:

Definition. [1.29]

Let us consider a given reference frame R_i endowed with its physical coordinate system $(o_i; a_i, b_i, c_i)$ and let

- O_i, A_i, B_i and C_i denote the respective positions of the particles o_i a_i, b_i and c_i, in R_0 and at the current instant t:

$$\boxed{O_i \equiv pos_{R_0}(o_i, t)} \quad \boxed{A_i \equiv pos_{R_0}(a_i, t) \quad B_i \equiv pos_{R_0}(b_i, t) \quad C_i \equiv pos_{R_0}(c_i, t)}$$

- and

$$\boxed{\vec{x}_i \equiv \overrightarrow{O_iA_i} \qquad \vec{y}_i \equiv \overrightarrow{O_iB_i} \qquad \vec{z}_i \equiv \overrightarrow{O_iC_i}}$$

R_i is said to be *endowed with the coordinate system* $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ or, put another way, $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ *is the coordinate system of* R_i. These expressions are convenient contractions in language that make it possible to designate the physical coordinate system of a reference frame using one point and three vectors, rather than four particles:

– "R_i is endowed with the coordinate system $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$" is a contraction of "$R_i$ is endowed with the physical coordinate system $(o_i; a_i, b_i, c_i)$ whose position in R_0 is $(O_i; A_i, B_i, C_i)$".

– "$(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ is the coordinate system of R_i" is the contraction of "$(O_i; A_i, B_i, C_i)$ is the position in R_0 of the physical coordinate system $(o_i; a_i, b_i, c_i)$ of R_i".

Note that the definition of a coordinate system of R_i involves the positions of the particles o_i, a_i, b_i, c_i in R_0.

When making figures, we usually plot the coordinate system $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ (more precisely, its position in R_0) to *represent* or *visualize* the reference frame R_i (see Figure 1.7).

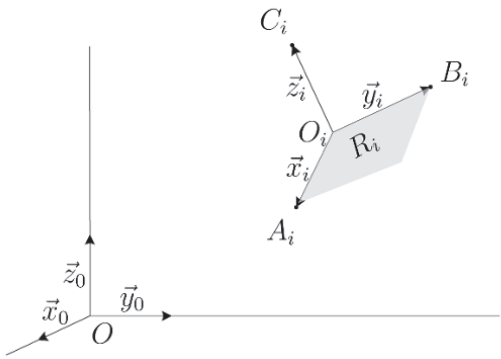


Figure 1.7. *Coordinate system $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ of reference frame R_i*

In the 2D example in Figure 1.6, $(O_2; \vec{x}_2, \vec{y}_2)$ is the coordinate system of R_2.

According to [1.28]–[1.28b], the coordinate system $(O_0; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ of the common reference frame R_0 is identified with the coordinate system $(O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ of the affine space $\mathcal{E}$:

$$\boxed{O_0 = O} \quad \text{and} \quad \boxed{(\vec{x}_0, \vec{y}_0, \vec{z}_0) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)} \qquad [1.30]$$

For any reference frame R_i, one has to be careful to avoid confusing the "coordinate system of R_i" and the "(mathematical) coordinate system":

– The coordinate system of R_i is indeed a coordinate system in the mathematical sense of the term (set of a point in $\mathcal{E}$ and of three vectors of E) and thus it can be given all the classical terminology relative to a mathematical coordinate system: the point O_i is called the *origin of the coordinate system of* R_i and $(\vec{x}_i, \vec{y}_i, \vec{z}_i)$ *the basis of the coordinate system of* R_i, the coordinate system of R_i is said to be *a right-handed orthonormal basis* if the basis $(\vec{x}_i, \vec{y}_i, \vec{z}_i)$ is right-handed orthonormal. We also speak of coordinates of a point in the coordinate system of R_i.

– Conversely, a mathematical coordinate system is not necessarily a coordinate system for R_i in the sense of the above definition. We will see, in [1.37], a condition required for the mathematical coordinate system to be the coordinate system of R_i.

Theorem. Let $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ be the coordinate system of a reference frame R_i, defined in [1.29]. The image of vectors $\vec{x}_i, \vec{y}_i, \vec{z}_i$ under $\bar{\bar{Q}}_{i0}$ are the vectors of the basis of E:

$$\boxed{(\bar{\bar{Q}}_{i0}.\vec{x}_i, \bar{\bar{Q}}_{i0}.\vec{y}_i, \bar{\bar{Q}}_{i0}.\vec{z}_i) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)} \qquad [1.31]$$

PROOF. Let us carry out the proof for $\vec{x}_i$, the proof for the two other vectors $\vec{y}_i, \vec{z}_i$ being similar. With the notations in definition [1.29] for the coordinate system $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ of R_i, we have:

$$\begin{aligned} \bar{\bar{Q}}_{i0}.\vec{x}_i &= \bar{\bar{Q}}_{i0}.\overrightarrow{O_iA_i} \\ &= \overrightarrow{O_i^{(i)}A_i^{(i)}} && \text{according to [1.25]} \\ &= \vec{e}_1 && \text{according to [1.14]}_{a-b} \quad \blacksquare \end{aligned}$$

In the case $R_i = R_0$, the relationship [1.31] gives

$$\boxed{(\vec{x}_0, \vec{y}_0, \vec{z}_0) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)} \qquad [1.32]$$

and we once again arrive at [1.30].

1.3.8. *Fixed point and fixed vector in a reference frame*

Let A be a point in $\mathcal{E}$. The expression "the point A is fixed in $\mathcal{E}$" is perfectly well defined. It signifies simply that at any instant A is the same point in $\mathcal{E}$. The following definition is a new definition:

Definition. Let A be a point in $\mathcal{E}$ (which does or does not vary with t). We say that A is *fixed in* R_i or *attached to* R_i if

$$\forall t, A^{(i)} \equiv Q_{i0}(t)A \text{ is a fixed point (in } \mathcal{E}) \qquad [1.33]$$

The following theorem ensures that definition [1.33] effectively renders the intuitive idea of the fixity of a point with respect to a reference frame.

Theorem. Let A be a point in $\mathcal{E}$ (which does or does not vary with t).
$A \in \mathcal{E}$ is fixed in $R_i \Leftrightarrow$, the (fictitious) particle of position A in R_0 at any instant is at a fixed position in R_i. [1.34]

PROOF. Let a denote the (fictitious) particle whose position in R_0 at any instant is A: $\forall t, A = pos_{R_0}(a, t)$ (there exists such a particle according to the position postulate [1.12]). The following equivalences hold:

$$\begin{aligned} A \text{ is fixed in } R_i &\Leftrightarrow \forall t, Q_{i0}A \text{ is a fixed point in } \mathcal{E} \text{ (definition [1.33])} \\ &\Leftrightarrow \forall t, Q_{i0}pos_{R_0}(a,t) \text{ is a fixed point in } \mathcal{E} \\ &\Leftrightarrow \forall t, pos_{R_i}(a,t) \text{ is a fixed point, according to the} \\ &\qquad \text{position change postulate [1.19]} \ \blacksquare \end{aligned}$$

Let us introduce the following definition, similar to [1.33]:

Definition. Let $\vec{W}$ be a vector in E (a vector that may or may not be variable with t). We say that $\vec{W}$ is *constant in* R_i (or *fixed in* R_i, or *attached to* R_i) if $\forall t, \vec{W}^{(i)} \equiv \bar{\bar{Q}}_{i0}(t)\vec{W}$ is a constant vector in E. [1.35]

The following theorem gives the physical interpretation for a constant vector in a reference frame:

Theorem. Let $\vec{W}$ be a vector in E (which does or does not vary with t).
$\vec{W} \in E$ is constant in $R_i \Leftrightarrow$ and
the biparticle of biposition $\vec{W}$ in R_0, at any instant, has a fixed biposition in R_i.

PROOF. Let us write $\vec{W} = B - A$, with $A, B \in \mathcal{E}$, a (respectively, b) the (fictitious) particle of position A (respectively, B) in R_0 at any instant: $\forall t, A = pos_{R_0}(a,t), B = pos_{R_0}(b,t)$.

According to definition [1.35], $\vec{W} \in E$ is fixed in $R_i \Leftrightarrow \forall t, \bar{\bar{Q}}_{10}(t)\vec{W}$ is a constant vector in E. Now,

$$\begin{aligned}
\bar{\bar{Q}}_{10}(t).\vec{W} &= \bar{\bar{Q}}_{10}.\overrightarrow{AB} \\
&= Q_{i0}B - Q_{i0}A \qquad \text{according to [1.21]} \\
&= Q_{i0}pos_{R_0}(b,t) - Q_{i0}pos_{R_0}(a,t) \\
&= pos_{R_i}(b,t) - pos_{R_i}(a,t) \text{ according to the position change postulate [1.19]} \blacksquare
\end{aligned}$$

REMARK. The fixity of a point and the constancy of a vector are concepts that have a sense only with respect to a reference frame. Moreover, one should not confuse the constancy of a vector with that of its norm. ■

The following theorem can be proved:

Theorem.

(i) The coordinate system $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ of a reference frame R_i, defined in [1.29], is *fixed in* R_i, that is

- the point O_i is a fixed point in R_i,
- and the vectors $\vec{x}_i, \vec{y}_i, \vec{z}_i$ are fixed vectors in R_i. [1.36]

(ii) Conversely, if a (mathematical) coordinate system $(O_i; \vec{x}_i, \vec{y}_i, \vec{z}_i)$ ($O_i \in \mathcal{E}$ and $\vec{x}_i, \vec{y}_i, \vec{z}_i \in E$) is fixed in a reference frame R_i, then it is a coordinate system of the reference frame R_i as defined in [1.29]. [1.37]

1.4. Derivative of a vector with respect to a reference frame

Consider a vector quantity (for example, the position vector of a particle) whose observation result in the common reference frame R_0 is a vector $\vec{W} \in E$ that is function of t (in R_0 we write $\vec{W}$ rather than $\vec{W}^{(0)}$).

The vector $\dfrac{d\vec{W}}{dt}$ ($\in E$) is the *classical* derivative of $\vec{W}$ with respect to time, a derivative that any observer can calculate by using universally known mathematics. One way of calculating $\dfrac{d\vec{W}}{dt}$ is to differentiate the components of $\vec{W}$ in a fixed basis of E.

Let us now introduce a new concept, more complicated than the previous classical derivative. It is called *the time derivative of the vector* $\vec{W}$ *with respect to a given reference frame* R_1.

Definition. *The time derivative of a vector* $\vec{W}$ with respect to a reference frame R_1, denoted by $\boxed{\dfrac{d_{R_1}\vec{W}}{dt} \text{ or, in abridged form } \dfrac{d_1\vec{W}}{dt}}$, is, by definition

$$\boxed{\frac{d_{R_1}\vec{W}}{dt} \equiv \bar{\bar{Q}}_{01}.\frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)} \in E \qquad [1.38]$$

The computation of $\dfrac{d_1\vec{W}}{dt}$ is made up of three elementary steps summarized in the flowchart

below:

$$\boxed{\begin{array}{ccc} \vec{W} \in E & & \dfrac{d_1\vec{W}}{dt} \in E \\ \downarrow (1) & & \uparrow (3) \\ \vec{W}^{(1)} \in E & \overset{(2)}{\rightarrow} & \dfrac{d\vec{W}^{(1)}}{dt} \in E \end{array}} \qquad [1.39]$$

This flowchart may be interpreted in figurative terms as follows. Let us imagine that the reference frames R_0 and R_1 are two infinite, transparent sheets of paper laid one atop the other, and sliding against each other, such that the observations in a reference frame may then be "*read through transparency and traced to the other reference frame*".

Assume that one can draw the vector under consideration, $\vec{W}$, on the sheet R_0 at different instants. At each instant t, one can take the vector $\vec{W}$ drawn on the sheet R_0 and "*trace it onto the sheet* R_1" to obtain the vector associated with $\vec{W}$ in R_1, $\vec{W}^{(1)} = \bar{\bar{Q}}_{10}.\vec{W}$. By repeating this tracing operation at different instants, it becomes possible to construct, on the sheet R_1, the family of vectors

$$t \mapsto \vec{W}^{(1)}(t) \qquad [1.40]$$

which is, *a priori*, *different* from the family of vectors $t \mapsto \vec{W}(t)$ drawn on the sheet R_0 because the two sheets R_0 and R_1 move with respect to one another. The mapping [1.40] represents the transfer onto R_1 of the evolution over time of the vector $\vec{W}$. This is operation (1) in flowchart [1.39].

Knowing the mapping [1.40], one calculates the derivative $\frac{d\vec{W}^{(1)}}{dt}$ of vector $\vec{W}^{(1)}$, which is simply the classical derivative with respect to time. Like $\vec{W}^{(1)}$, the derivative $\frac{d\vec{W}^{(1)}}{dt}$ is drawn on the sheet R_1. This is operation (2) in flowchart [1.39].

The third and final operation, number (3) in the flowchart (i.e. calculating $\bar{\bar{Q}}_{01}.\frac{d\vec{W}^{(1)}}{dt}$), consists of "*tracing* $\frac{d\vec{W}^{(1)}}{dt}$ *backwards onto the sheet* R_0" to obtain vector $\frac{d_{R_1}\vec{W}}{dt}$ defined in [1.38]. This operation is required in rigid bodies mechanics with the objective of simplifying the formulae obtained in kinematics.

• For a vector $\vec{W}$ such that $\bar{\bar{Q}}_{10}.\vec{W}$ depends on time and other space variables, the partial derivative of $\vec{W}$ with respect to time in the reference frame R_1 is defined in a similar manner:

Definition. *The time partial derivative of a vector* $\vec{W}$ *with respect to a reference frame* R_1, denoted by $\boxed{\dfrac{\partial_{R_1}\vec{W}}{\partial t}}$, is, by definition

$$\boxed{\frac{\partial_{R_1}\vec{W}}{\partial t} \equiv \bar{\bar{Q}}_{01}.\frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)} \in E \qquad [1.41]$$

Theorem.

$$\frac{d_{R_1}\vec{W}}{dt} = \vec{0} \qquad \Leftrightarrow \qquad \vec{W} \text{ is fixed in } R_1 \ \ (\text{see definition [1.35]}) \qquad [1.42]$$

DEMONSTRATION.

$$\frac{d_{R_1}\vec{W}}{dt} = \vec{0} \Leftrightarrow \frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\vec{W}\right) = \vec{0} \qquad \text{according to definition [1.38]}$$
$$\Leftrightarrow \bar{Q}_{10}.\vec{W} \text{ is a constant vector in } E \qquad \blacksquare$$

We also have the following classical relationships (whose demonstration can be found in books on Newtonian mechanics):

Theorem. $\forall$ reference frame $R_1, \forall \vec{U}, \vec{V}, \vec{W} \in E, \forall \lambda \in \mathbb{R}$,

$$\frac{d_{R_1}}{dt}(\vec{U}+\vec{V}) = \frac{d_{R_1}\vec{U}}{dt} + \frac{d_{R_1}\vec{V}}{dt} \qquad [1.43]$$

$$\frac{d_{R_1}}{dt}(\lambda\vec{W}) = \frac{d\lambda}{dt}\vec{W} + \lambda\frac{d_{R_1}\vec{W}}{dt} \qquad [1.44]$$

$$\frac{d}{dt}(\vec{U}.\vec{V}) = \frac{d_{R_1}\vec{U}}{dt}.\vec{V} + \vec{U}.\frac{d_{R_1}\vec{V}}{dt} \qquad [1.45]$$

Note that the derivative on the left-hand side of [1.45] is the derivative of a scalar function and does not depend on the reference frame, while the derivatives on the right-hand side relate to vectors and carry the index for the reference frame R_1.

Theorem. Consider $\vec{W} = \alpha\vec{x}_1 + \beta\vec{y}_1 + \gamma\vec{z}_1 \in E$, where $b_1 = (\vec{x}_1, \vec{y}_1, \vec{z}_1)$ is a vector basis in E, *fixed in* R_1. Then

$$\boxed{\frac{d_{R_1}\vec{W}}{dt} = \dot{\alpha}\vec{x}_1 + \dot{\beta}\vec{y}_1 + \dot{\gamma}\vec{z}_1} \qquad [1.46]$$

PROOF. Apply relationships [1.42]–[1.44]. ■

1.5. Velocity of a particle

Definition. By definition, *the velocity, with respect to the reference frame* R_1 *and at the instant* t, *of a particle* p whose position is P is

$$\boxed{\vec{V}_{R_1}(p,t) = \frac{d_{R_1}\overrightarrow{O_1P}}{dt}} \underset{[1.38]}{=} \bar{\bar{Q}}_{01}.\frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right) \qquad [1.47]$$

where O_1 is a *fixed point in* R_1.

1.6. Angular velocity

The following theorem is a classic result in Newtonian mechanics of rigid bodies:

Theorem and definition. Composite derivative of a vector. Let R_1, R_2 be two reference frames. We have: $\forall$ vector $\vec{W} \in E$:

$$\boxed{\frac{d_1\vec{W}}{dt} = \frac{d_2\vec{W}}{dt} + \vec{\Omega}_{12} \times \vec{W}} \text{ with } \boxed{\vec{\Omega}_{12} \equiv \vec{\Omega}_{R_1R_2} \equiv \frac{1}{2}\sum_{j=1}^{3}\vec{b}_j \times \frac{d_1\vec{b}_j}{dt}}, \qquad [1.48]$$

where $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ is an orthonormal basis made of vectors in E, *fixed in* R_2 (Figure 1.8). The vector $\vec{\Omega}_{12}$ is called *the angular velocity vector of* R_2 *with respect to* R_1 *(at instant* t*)*.

The *skew-symmetric* tensor $\bar{\bar{\Omega}}_{12}$ associated with $\vec{\Omega}_{12}$ is called the *angular velocity tensor of* R_2 *with respect to* R_1 *(at instant* t*)*. It is related to the rotation tensors $\bar{\bar{Q}}_{01}, \bar{\bar{Q}}_{02}$ and $\bar{\bar{Q}}_{12}$ through

$$\boxed{\bar{\bar{\Omega}}_{12} = \bar{\bar{Q}}_{01}.\frac{d\bar{\bar{Q}}_{12}}{dt}.\bar{\bar{Q}}_{20}} \qquad [1.49]$$

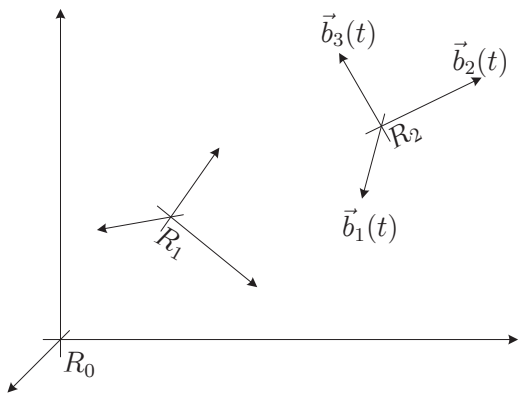


Figure 1.8. *Composite derivative of a vector*

To obtain an explicit expression for $\vec{\Omega}_{12}$ in [1.48], let us write $[\vec{x}_2, \vec{y}_2, \vec{z}_2]$ instead of $[\vec{b}_1, \vec{b}_2, \vec{b}_3]$. We then have

$$\boxed{\vec{\Omega}_{12} = \frac{1}{2}\left(\vec{x}_2 \times \frac{d_1\vec{x}_2}{dt} + \vec{y}_2 \times \frac{d_1\vec{y}_2}{dt} + \vec{z}_2 \times \frac{d_1\vec{z}_2}{dt}\right)}$$

The following theorem is an important particular case of [1.48] :

Theorem. Derivative formula with respect to R_1 of a vector constant in R_2.

$$\forall R_1, R_2, \forall \vec{W} \in E \text{ constant in } R_2, \boxed{\frac{d_1\vec{W}}{dt} = \vec{\Omega}_{12} \times \vec{W}} \qquad [1.50]$$

PROOF. One has just to apply [1.48], noting that as $\vec{W}$ is constant in R_2, we have $\frac{d_2\vec{W}}{dt} = \vec{0}$ according to [1.42]. ■

Theorem. Composition of angular velocities.

$$\forall R_1, R_2, R_3, \quad \boxed{\vec{\Omega}_{13} = \vec{\Omega}_{12} + \vec{\Omega}_{23}} \qquad [1.51]$$

PROOF. Let us apply [1.48] to three pairs of reference frames (R_1, R_2), (R_2, R_3) and (R_1, R_3):

$$\forall \vec{W} \in E, \frac{d_1\vec{W}}{dt} = \frac{d_2\vec{W}}{dt} + \vec{\Omega}_{12} \times \vec{W} \qquad \frac{d_2\vec{W}}{dt} = \frac{d_3\vec{W}}{dt} + \vec{\Omega}_{23} \times \vec{W} \qquad \frac{d_1\vec{W}}{dt} = \frac{d_3\vec{W}}{dt} + \vec{\Omega}_{13} \times \vec{W} \qquad [1.52]$$

Adding each side of the first two equalities leads to

$$\frac{d_1\vec{W}}{dt} = \frac{d_3\vec{W}}{dt} + (\vec{\Omega}_{12} + \vec{\Omega}_{23}) \times \vec{W} \qquad [1.53]$$

By identifying [1.53] with the last equality of [1.52], we find: $\vec{\Omega}_{13} \times \vec{W} = (\vec{\Omega}_{12} + \vec{\Omega}_{23}) \times \vec{W}$, hence [1.51] taking into account the fact that vector $\vec{W}$ is arbitrary. ■

By making $R_3 = R_1$ in [1.51], we can immediately deduce:

Corollary. The angular velocity of R_2 with respect to R_1 is the opposite of the angular velocity of R_1 with respect to R_2:

$$\boxed{\vec{\Omega}_{12} = -\vec{\Omega}_{21}} \qquad [1.54]$$

1.7. Reference frame defined by a rigid body: Rigid body defined by a reference frame

Before discussing the kinematics of rigid bodies, let us introduce two terminologies that will be necessary for what follows.

In section 1.3, we saw that a reference solid is a support that makes it possible for an observer to locate particles at each instant in the physical space. We admit that a reference solid is indeed a rigid body (a solid) in the classic sense of rigid body mechanics. The following definition is easy to understand:

Definition. *The reference frame defined by a rigid body* S is the reference frame whose reference solid is S. It is often denoted by $\boxed{R_S}$. We also say that the *rigid body* S *defines the reference frame* R_S. [1.55]

By agreeing on the fact that a particle p is *fixed in a reference frame* R_1 if $\forall t$, its position $P^{(1)}(t) = pos_{R_1}(p,t)$ in R_1 is the same point of $\mathcal{E}$, we will now introduce the reciprocal concept for [1.55], namely, the rigid body defined by a reference frame:

Definition. *The rigid body defined by a reference frame* R_1, denoted by $\boxed{S(R_1)}$, is, by definition, the union of all fixed particles in R_1 (which is indeed a rigid body). [1.56]

From the very definition of $S(R_1)$, we have the following equivalence:

$$p \text{ is a fixed particle in } R_1 \Leftrightarrow \text{ is a particle of } S(R_1) \text{ (i.e., } p \subset S(R_1))$$

As all the particles of the reference solid in R_1 are fixed in R_1, *the reference solid of* R_1 *is a subset of* $S(R_1)$. We say that $S(R_1)$ "*extends the reference solid of* R_1 *to infinity*".

For all reference frames R_1, R_2 and at any instant t, $pos_{R_2}(S(R_1), t)$ is the entire space $\mathcal{E}$.

1.8. Point attached to a rigid body: Vector attached to a rigid body

In section 1.3.8, we defined the concepts of a point and a vector attached to a given reference frame R_i. Given a rigid body S, one can define its reference frame R_S and talk of a point or vector attached to R_S. We will now introduce two new concepts, namely the point attached to S and the vector attached to S.

Definition. [1.57]

A point A in $\mathcal{E}$ (that does or not vary with t) is *attached to the rigid body S* or *"follows" a particle of S*

– if A is the position in R_0 *at all instants* t of a particle *of* S,

– in other words, if there exists a particle a *of* S such that $\forall t, pos_{R_0}(a,t) = A$.

We can show the equivalence between the concept of a fixed point in a reference frame (definition [1.33]) and the concept of a point attached to a rigid body (definition [1.57] above):

Theorem.

$$A \text{ is attached to } S \Leftrightarrow A \text{ is attached to } R_S \qquad [1.58]$$

Thus, the two expressions "A is attached to S" and "A is attached to R_S" are *synonymous*, and the two definitions [1.57] and [1.33] are consistent.

• The concept of a vector attached to a rigid body is defined in the same way:

Definition. [1.59]

A vector $\vec{W}$ in E (a vector that may or may not vary with t) is *attached to the rigid body S*

– if $\vec{W}$ is the biposition of a biparticle *of* S in R_0 *at any instant* t,

– in other words, if there exist two particles a, b *of* S such that $\forall t, pos_{R_0}(b,t) - pos_{R_0}(a,t) = \vec{W}$.

The following equivalence can be proved in a manner similar to [1.58]:

Theorem.

$$\vec{W} \text{ is attached to } S \Leftrightarrow \vec{W} \text{ is attached to } R_S \qquad [1.60]$$

Thus, the two expressions "$\vec{W}$ is attached to S" and "$\vec{W}$ is attached to R_S" are *synonymous*, and the two definitions [1.59] and [1.35] are consistent.

• In the presence of a reference frame R_S defined by a rigid body S, we can interchangeably use "point (respectively, vector) attached to R_S" or "point (respectively, vector) attached to S", with a preference for the latter expression, whose physical significance is more direct.

1.9. Velocities in a rigid body

The results in this section are valid for *a rigid body*.

Theorem and definition. $\forall t$, $\forall$ reference frame R_1, $\forall$ *rigid body* S defining a reference frame R_S, $\forall$ particles p, p' belonging to the rigid body S, whose positions in R_0 are, respectively, P and P', we have

$$\boxed{\vec{V}_{R_1}(p',t) = \vec{V}_{R_1}(p,t) + \vec{\Omega}_{R_1 R_S} \times \overrightarrow{PP'}}, \qquad [1.61]$$

where $\vec{\Omega}_{R_1 R_S}$ is the angular velocity vector of R_S with respect to R_1, as defined in [1.48].

PROOF. Let O_1 be a point of $\mathcal{E}$, fixed in R_1. We have

$$\vec{V}_{R_1}(p',t) - \vec{V}_{R_1}(p,t) = \frac{d_1}{dt}\overrightarrow{O_1P'} - \frac{d_1}{dt}\overrightarrow{O_1P} = \frac{d_1}{dt}\overrightarrow{P(t)P'(t)}$$

As the particles p, p' belong to the rigid body S, their positions in R_S, $P^{(S)}$ and $P'^{(S)}$ are fixed points in $\mathcal{E}$ over time and, therefore, $\forall t$, $\overrightarrow{P^{(S)}P'^{(S)}}$ is a constant vector in E. Further, according to [1.22], we have $\forall t$, $\overrightarrow{P^{(S)}P'^{(S)}} = \overline{\overline{Q}}_{S0}.\overrightarrow{PP'}$, where $\overline{\overline{Q}}_{S0}.\overrightarrow{PP'}$ is a constant vector of E. In other words, the vector $\overrightarrow{PP'}$ is constant in R_S according to the definition [1.35].

We thus have $\dfrac{d_1}{dt}\overrightarrow{P(t)P'(t)} = \vec{\Omega}_{R_1R_S} \times \overrightarrow{P(t)P'(t)}$ on application of theorem [1.50]. ■

Velocity field

The notation $\vec{V}_{R_1}(p,t)$ is quite natural and easy to understand. The only problem it poses is that it contains a particle as an argument, which does not make it possible to talk of velocity *fields*, whose arguments are *points* in the affine space $\mathcal{E}$. The following, new, notation, called "Eulerian" notation, is a little more complex but enables us to resolve this problem.

Eulerian (or spatial) notation. Let S be a rigid body and let there be a point $A \in pos_{R_0}(S,t)$. We write

$$\boxed{\vec{V}_{R_1S}(A,t) \equiv \begin{array}{l}\text{the velocity with respect to } R_1 \text{ and at instant } t \\ \text{of the particle of } S \text{ passing through point } A \text{ at instant } t\end{array}} \qquad [1.62]$$

When using the Eulerian notation, the particle is not known by its name but by its position at the instant considered. In general, the particle is *not* the same over the course of time.

As opposed to $\vec{V}_{R_1S}(A,t)$, the notation $\vec{V}_{R_1}(p,t)$ is said to be Lagrangian (or material). To illustrate the difference between the two velocities, Lagrangian and Eulerian, imagine that the reference frame R_1 is defined by the rails on which a train, S, is running. The Lagrangian description consists of observing, over time, a given particle p of the train, materialized, for example, by a corner of a window on the train. As the train moves, this particle will change position with respect to the ground and its velocity is denoted by $\vec{V}_{R_1}(p,t)$. With the Eulerian description, imagine an electric pole positioned in the ground along the tracks. Look at the shadow of the tip of the pole that falls upon the train, which is a *fixed* point A with respect to R_1. At a given instant t, the Eulerian velocity $\vec{V}_{R_1S}(A,t)$ is the velocity of the particle of the train that passes through the point A at this instant. As the point A is fixed, the particle that passes through A is *not* the same over time, contrary to the case of Lagrangian velocity.

In general, Lagrangian and Eulerian velocities are not identical. One may prove to be preferable to the other or may even prove indispensable, depending on the context of the mechanical problem.

• Let p be a particle of S located at $P = pos_{R_0}(p,t)$ in R_0 over time. We have the trivial equality

$$\forall t, \qquad \vec{V}_{R_1S}(P,t) = \vec{V}_{R_1}(p,t) \overset{[1.47]}{=} \frac{d_{R_1}\overrightarrow{O_1P}}{dt}$$

On the contrary, for any point $A \in \mathcal{E}$:

$$\vec{V}_{R_1S}(A,t) \neq \frac{d_{R_1}\overrightarrow{O_1A}}{dt}$$

The equality holds only when A is a point *attached to the rigid body* S, that is, when A denotes the position of the *same* particle of the rigid body over time.

• Using the Eulerian notation [1.62], we can define *the velocity field of a rigid body* S with respect to R_1 and at the instant t:

$$\begin{array}{rcl} V_{R_1S}(.,t): \ \mathcal{E} \supset pos_{R_0}(S,t) & \to & E \\ A & \mapsto & \vec{V}_{R_1S}(A,t) \end{array}$$

Velocity field in a rigid body

Theorem. $\forall t, \forall R_1, \forall$ *rigid body* S defining a reference frame R_S, $\forall A, B \in pos_{R_0}(S,t) \subset \mathcal{E}$,

$$\boxed{\vec{V}_{R_1S}(B,t) = \vec{V}_{R_1S}(A,t) + \vec{\Omega}_{R_1R_S} \times \overrightarrow{AB}} \qquad [1.63]$$

Thus, $\forall t$, the velocity field $V_{R_1S}(.,t)$ (defined on $pos_{R_0}(S,t)$) is completely determined by the velocity at one point (here, point A) and the angular velocity $\vec{\Omega}_{R_1R_S}$.

PROOF. Let t be a fixed instant, and A, B two given points $\in pos_{R_0}(S,t) \subset \mathcal{E}$. *At the instant* t,

- the point A is the position $P(q,t)$ of a particle p of the rigid body S with respect to R_0;
- the point B is the position $P'(q,t)$ of a particle p' of the rigid body S with respect to R_0.

We then have *at the instant* t

$$\vec{V}_{R_1S}(A,t) = \vec{V}_{R_1}(p,t) \qquad \vec{V}_{R_1S}(B,t) = \vec{V}_{R_1}(p,t') \qquad [1.64]$$

The proof is achieved by applying [1.61] at the instant t and with the particles p, p' of S that were just defined. ■

The equalities [1.64] are only valid at the instant t, but this suffices for the proof.

For mathematical convenience, we may define the velocity field over the entire space $\mathcal{E}$ and not only over $pos_{R_0}(S,t)$. To do this, we must carry out a classical operation in rigid bodies mechanics, which consists of extending the rigid body S "to infinity", so as to be able to state the previous theorem with the velocity field of a rigid body S defined over all of $\mathcal{E}$. However, we have not done this.

In practice, we write relationship [1.63] in abridged form as follows:

$$\forall t, \forall R_1, \forall \text{ rigid body } S, \forall A, B \in pos_{R_0}(S,t) \subset \mathcal{E}: \boxed{\vec{V}_{1S}(B,t) = \vec{V}_{1S}(A,t) + \vec{\Omega}_{1S} \times \overrightarrow{AB}}$$

1.10. Velocities in a mechanical system

We can easily generalize the Eulerian notation [1.62] to a mechanical system:

Eulerian notation. Let $\mathcal{S}$ be a mechanical system and A be a point such that $A \in pos_{R_0}(\mathcal{S},t)$. We write

$$\boxed{\begin{array}{rcl} \vec{V}_{R_1\mathcal{S}}(A,t) & \equiv & \text{the velocity with respect to } R_1 \text{ and at the instant } t \\ & & \text{of the particle of } \mathcal{S} \text{ passing through point } A \text{ at instant } t \end{array}} \qquad [1.65]$$

When using the Eulerian notation, the particle is not known by its name but is known by its position at the instant considered. In general, the particle is *not* the same over time.

There exists a case where the notation $\vec{V}_{R_1\mathcal{S}}(A,t)$ is ambiguous. This is when A is the point of contact I between two rigid bodies S_i and S_j in $\mathcal{S}$ (Figure 1.9). In this case, we must distinguish between two velocities $\vec{V}_{R_1S_i}(I,t)$ and $\vec{V}_{R_1S_j}(I,t)$ for two particles of S_i and S_j respectively, passing through the same point I at the instant considered t. These particles are infinitely close but are not identical.

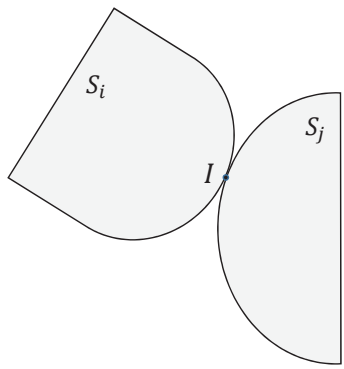


Figure 1.9. *Velocity at the contact point between two rigid bodies*

In general, the point I is attached neither to S_i nor to S_j and

$$\vec{V}_{R_1S_i}(I,t) \neq \vec{V}_{R_1S_j}(I,t)$$

We see the importance of the second index, indicating the rigid body in the notation for the velocity vector.

• Let p be a particle of $\mathcal{S}$ located at $P = pos_{R_0}(p,t)$ in R_0 over time. We then have the trivial equality

$$\forall t, \qquad \vec{V}_{R_1\mathcal{S}}(P,t) = \vec{V}_{R_1}(p,t) \overset{[1.47]}{=} \frac{d_{R_1}\overrightarrow{O_1P}}{dt}$$

On the contrary, for any point $A \in \mathcal{E}$, which is not attached to the system $\mathcal{S}$:

$$\vec{V}_{R_1\mathcal{S}}(A,t) \neq \frac{d_{R_1}\overrightarrow{O_1A}}{dt}$$

This equality holds only when A is a point *attached to system* $\mathcal{S}$, i.e. when A denotes the position of the *same* particle in the system over time.

• Using the Eulerian notation [1.65], we can define *the velocity field of a system* $\mathcal{S}$ with respect to R_1 and at an instant t:

$$\begin{array}{rcl} V_{R_1\mathcal{S}}(.,t): \ \mathcal{E} \supset pos_{R_0}(\mathcal{S},t) & \rightarrow & E \\ A & \mapsto & \vec{V}_{R_1\mathcal{S}}(A,t) \end{array}$$

1.11. Acceleration

1.11.1. *Acceleration of a particle*

Definition. *The acceleration of a particle p with respect to the reference frame R_1 and at the instant t* is, by definition,

$$\boxed{\vec{\Gamma}_{R_1}(p,t) = \frac{d_{R_1}\vec{V}_{R_1}(p,t)}{dt}} \qquad [1.66]$$

1.11.2. *Accelerations in a mechanical system*

For accelerations, we introduce analogous to [1.65] the following notation:

Eulerian notation. Let $\mathcal{S}$ be a mechanical system and A be a point such that $A \in pos_{R_0}(\mathcal{S},t)$. We write

$$\boxed{\vec{\Gamma}_{R_1\mathcal{S}}(A,t) \equiv \begin{array}{l}\text{the acceleration with respect to } R_1 \text{ and at the instant } t \\ \text{of the particle of } \mathcal{S} \text{ passing through point } A \text{ at instant } t\end{array}} \qquad [1.67]$$

1.12. Composition of velocities and accelerations

Before concluding this chapter, we recall the well-known results in kinematics and we introduce some neologisms, which will be used sometimes in the rest of the book.

1.12.1. *Composition of velocities*

Theorem. $\forall R_1, R_2, \forall$ system $\mathcal{S}, \forall A \in \mathcal{E}$:

$$\boxed{\vec{V}_{R_1\mathcal{S}}(A) = \vec{V}_{R_1S(R_2)}(A) + \vec{V}_{R_2\mathcal{S}}(A)} \quad \text{or shortly: } \boxed{\vec{V}_{1\mathcal{S}}(A) = \vec{V}_{12}(A) + \vec{V}_{2\mathcal{S}}(A)} \qquad [1.68]$$

If system $\mathcal{S}$ is a rigid body S, then an immediate consequence of [1.68] is

$$\boxed{\vec{\Omega}_{1S} = \vec{\Omega}_{12} + \vec{\Omega}_{2S}} \qquad [1.69]$$

which is merely [1.51].

The motion of $\mathcal{S}$ with respect to reference frame R_1 is called *the absolute motion*. The motion of $\mathcal{S}$ with respect to reference frame R_2 is called *the relative motion*. Similarly, velocities $\vec{V}_{R_1\mathcal{S}}(A)$ and $\vec{V}_{R_2\mathcal{S}}(A)$ are called *the absolute velocity* and *the relative velocity*, respectively.

According to notations [1.56] and [1.65], $\vec{V}_{R_1S(R_2)}(A) \equiv \vec{V}_{12}(A)$ is the velocity with respect to R_1 of the particle of rigid body $S(R_2)$ (that is, the particle attached to R_2), which coincides with A at the instant t. To refer to this kind of velocity, we introduce the following new term:

Definitions. [1.70]

- The motion of $S(R_2)$ (or simply, of R_2) with respect to reference frame R_1 is called *the background motion*.
- The velocity $\vec{V}_{R_1S(R_2)}(A) \equiv \vec{V}_{12}(A)$ is called *the background velocity*.

With this terminology, relation [1.68] means that the absolute velocity is equal to the sum of the relative velocity and the background velocity.

The absolute motion is the composition of the relative motion and the motion underlying it, namely the background motion.

1.12.2. *Composition of accelerations*

Theorem. $\forall R_1, R_2, \forall$ system $\mathcal{S}, \forall A \in \mathcal{E}$:

$$\boxed{\vec{\Gamma}_{R_1\mathcal{S}}(A) = \vec{\Gamma}_{R_1 S(R_2)}(A) + \vec{\Gamma}_{R_2\mathcal{S}}(A) + 2\vec{\Omega}_{R_1R_2} \wedge \vec{V}_{R_2\mathcal{S}}(A)}$$

$$\text{or shortly: } \boxed{\vec{\Gamma}_{1\mathcal{S}}(A) = \vec{\Gamma}_{12}(A) + \vec{\Gamma}_{2\mathcal{S}}(A) + 2\vec{\Omega}_{12} \wedge \vec{V}_{2\mathcal{S}}(A)} \qquad [1.71]$$

Accelerations $\vec{\Gamma}_{R_1\mathcal{S}}(A)$ and $\vec{\Gamma}_{R_2\mathcal{S}}(A)$ are called *the absolute acceleration* and *the relative acceleration*, respectively. The term $2\vec{\Omega}_{R_1R_2} \times \vec{V}_{R_2\mathcal{S}}(A)$ is *the Coriolis acceleration*. The following new definition is analogous to [1.70]:

Definition. The acceleration $\vec{\Gamma}_{R_1S(R_2)}(A) \equiv \vec{\Gamma}_{12}(A)$ is called *the background acceleration*. [1.72]

1.13. Angular momentum: Dynamic moment

Consider a given reference frame R_1 and a system $\mathcal{S}$. A current particle of the system is denoted by p, its position with respect to R_0 is $P(t) = pos_{R_0}(p,t)$.

The momentum, or *quantity of movement*, is a vector equal to a mass times a velocity, thus having the form $m\vec{V}$ or $\vec{V}dm$.

The angular momentum of system $\mathcal{S}$ about a point A (with respect to R_1 and at instant t), denoted by $\vec{\sigma}_{R_1\mathcal{S}}(A,t)$, is defined as the moment of the quantities of movement about point A:

$$\boxed{\vec{\sigma}_{R_1\mathcal{S}}(A,t) \equiv \int_{\mathcal{S}} \overrightarrow{AP}(t) \wedge \vec{V}_{R_1}(p,t)dm = \int_{\mathcal{S}} \overrightarrow{AP}(t) \wedge \vec{V}_{R_1\mathcal{S}}(P,t)dm} \qquad [1.73]$$

In the first integral above, the integration variable is particle $p \subset \mathcal{S}$, whereas in the second integral the integration variable is point P belonging to the position $pos_{R_0}(\mathcal{S},t)$ of the system with respect to R_0.

The angular momentum of system $\mathcal{S}$ about an axis Δ (with respect to R_1 and at instant t), denoted by $\sigma_{R_1\mathcal{S}}(\Delta,t)$, is defined as the projection of the angular momentum about any point A belonging to Δ onto the unit vector $\vec{u}$ parallel to Δ:

$$\boxed{\sigma_{R_1\mathcal{S}}(\Delta,t) \equiv \vec{\sigma}_{R_1\mathcal{S}}(A,t).\vec{u}} \qquad [1.74]$$

The right-hand side does not actually depend on the point A chosen on Δ.

The following new terms are defined in exactly the same way as above, with the velocity replaced by the acceleration:

Definitions.

– *The quantity of acceleration* is a vector equal to a mass times an acceleration, thus having the form $m\vec{\Gamma}$ or $\vec{\Gamma}dm$.

– *The dynamic moment* of system $\mathcal{S}$ about point A (with respect to R_1 and at instant t), denoted by $\vec{\delta}_{R_1\mathcal{S}}(A,t)$, is defined as the moment of the quantities of acceleration:

$$\boxed{\vec{\delta}_{R_1\mathcal{S}}(A,t) = \int_{\mathcal{S}} \overrightarrow{AP}(t) \wedge \vec{\Gamma}_{R_1}(p,t)dm \equiv \int_{\mathcal{S}} \overrightarrow{AP}(t) \wedge \vec{\Gamma}_{R_1\mathcal{S}}(P,t)dm} \qquad [1.75]$$

– *The dynamic moment of system $\mathcal{S}$ about axis* Δ (with respect to R_1 and at instant t), denoted by $\delta_{R_1\mathcal{S}}(\Delta,t)$, is defined as the projection of the dynamic moment about any point A belonging to Δ onto the unit vector $\vec{u}$ parallel to Δ:

$$\boxed{\delta_{R_1\mathcal{S}}(\Delta,t) \equiv \vec{\delta}_{R_1\mathcal{S}}(A,t).\vec{u}} \qquad [1.76]$$

The right-hand side does not actually depend on the point A chosen on Δ.

2

Parameterization and Parameterized Kinematics

The kinematics results reviewed in Chapter 1 are results that are also known in Newtonian mechanics. In this chapter, we will introduce the following concepts that are (except for the first concept) specific to analytical mechanics:

1. the position parameters of a mechanical system;
2. the mechanical joints and the constraint equations expressed as relationships between position parameters;
3. the parameterization, that is, the manner in which the constraint equations of the problem are categorized and, consequently, how the position parameters are classified into primitive and retained parameters;
4. some additional results on velocity, taking into account the chosen parameterization;
5. the parameterized velocities and Lagrange kinematic formulae, which will help the reader prepare for the Lagrange's equations which will be studied in the following chapter.

2.1. Position parameters

2.1.1. *Position parameters of a particle*

Consider a particle p. Its position $P = pos_{R_0}(p,t)$, in the reference frame R_0 and at a given instant t, is a point in $\mathcal{E}$.

Let the reference frame R_0 be endowed with an orthonormal coordinate system $(O;\vec{x}_0,\vec{y}_0,\vec{z}_0) = (O;\vec{e}_1,\vec{e}_2,\vec{e}_3)$ (see [1.30]). The point P may be defined, *a priori*, by its three Cartesian, cylindrical or spherical coordinates in the coordinate system in question. The expression "*a priori*" signifies that at this stage we consider the particle to be *free* in space, without taking into account any possible constraints between this particle and other bodies (these constraints will be discussed in section 2.2).

Consider, for example, a particle moving in the plane $O\vec{x}_0\vec{y}_0$, and the *a priori* position P of the particle may be defined by two Cartesian coordinates.

In certain cases, the position of the particle may also depend on time.

EXAMPLE. Consider a particle whose position is P, moving in a plane endowed with an orthonormal coordinate system $(O;\vec{x}_0,\vec{y}_0)$ (Figure 2.1). Its position may be defined by the two Cartesian coordinates x, y of P relative to $(O;\vec{x}_0,\vec{y}_0)$:

$$\overrightarrow{OP} = x\vec{x}_0 + y\vec{y}_0 \qquad \text{or } P = P(x,y) \qquad [2.1]$$

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

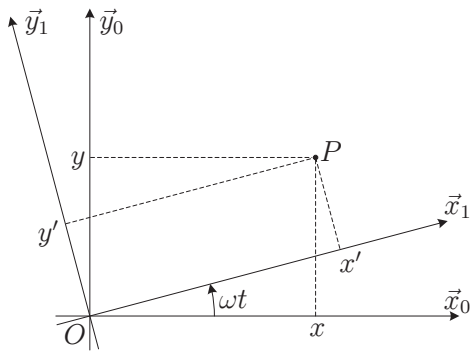


Figure 2.1. *Particle moving in a plane*

In certain problems, we may decide to define the position of the particle in a different way, introducing an intermediate basis $(\vec{x}_1, \vec{y}_1)$ that rotates at the constant angular velocity ω about $\vec{z}_0$. If x', y' denote the Cartesian coordinates of P relative to the rotating coordinate system, the position P of the particle may be expressed as a function of the parameters x', y', t as follows:

$$\overrightarrow{OP} = x'\vec{x}_1 + y'\vec{y}_1 = (x' \cos\omega t - y' \sin\omega t)\vec{x}_0 + (x' \sin\omega t + y' \cos\omega t)\vec{y}_0$$

that is

$$P = P(x', y', t) \qquad [2.2]$$

According to this point of view, the position of the particle depends explicitly on time t. The two points of view, [2.1] and [2.2], are equivalent. The choice of the point of view is dictated by the context of the problem to be solved. ■

2.1.2. *Position parameters for a rigid body*

As defined in [1.15], the position $pos_{R_0}(S, t)$ of a rigid body S, in the reference frame R_0 and at an instant t, is the set of positions P, at t, for all particles of S.

If the rigid body is not rectilinear, its *a priori* position may be defined by the three coordinates of the position of a particle of the rigid body and three angles. As is the case for a particle, the expression "*a priori*" signifies that at this stage, we consider the rigid body to be *free* in space, without taking into account any possible mechanical joints with other bodies.

Let us review the *Euler angles*, which are most widely used in rigid body mechanics to define the angular position of a rigid body.

EXAMPLE. EULER ANGLES.

Consider a non-rectilinear rigid body S moving in the physical space. Let the common reference frame R_0 be endowed with the right-handed orthonormal coordinate system $(O; b_0)$ where $b_0 \equiv (\vec{x}_0, \vec{y}_0, \vec{z}_0)$.

As the rigid body S is not rectilinear, we may choose four non-coplanar particles within this, o_S, a_S, b_S, c_S, such that the physical coordinate system $(o_S; a_S, b_S, c_S)$ is physically orthonormal in the sense of definition [1.9] (if S is planar, it can always be extended, in a unique way, into the

3D space, and we then can choose three particles in the plane and the fourth particle outside the plane). Let us denote:

$$O_S \equiv pos_{R_0}(o_S, t) \qquad A_S \equiv pos_{R_0}(a_S, t) \qquad B_S \equiv pos_{R_0}(b_S, t) \qquad C_S \equiv pos_{R_0}(c_S, t)$$
$$\vec{x}_S \equiv \overrightarrow{O_S A_S} \qquad \vec{y}_S \equiv \overrightarrow{O_S B_S} \qquad \vec{z}_S \equiv \overrightarrow{O_S C_S}$$

The basis $b_S \equiv (\vec{x}_S, \vec{y}_S, \vec{z}_S)$ thus constructed is a right-handed orthonormal basis. The point O_S and the basis b_S are attached to S. It is assumed that there is no constraint between b_0 and b_S (we say that the rotation of the rigid body S relative to R_0 is *free*).

The position in R_0 of the rigid body S is known as soon as we know the three coordinates of the point O_S relative to the coordinate system $(O; b_0)$ and the position of the basis b_S relative to the basis b_0.

We will show that the position of b_S relative to b_0 may be determined using three angles. To this end, let us introduce two intermediate bases between b_S and b_0.

Definition for the Euler angles and the intermediate bases u, v [2.3]

1. The first intermediate basis, denoted by $\boxed{u \equiv (\vec{n}, \vec{u}, \vec{z}_0)}$, is defined as follows:

   - Consider a vector straight line that is orthogonal to both $\vec{z}_0$ and $\vec{z}_S$ (there only exists one such line if $\vec{z}_0$ and $\vec{z}_S$ are non-collinear. If not, there exist an infinity of such lines) (Figure 2.2). In the frequent case when $\vec{z}_0$ and $\vec{z}_S$ are not collinear, this line is also the intersection between the vector planes $\vec{x}_S\vec{y}_S$ and $\vec{x}_0\vec{y}_0$.

     We arbitrarily choose one of the two unit vectors that orient this straight line and denote it by $\vec{n}$.
   - We then define the unit vector $\vec{u} \equiv \vec{z}_0 \times \vec{n}$.

   The basis $u \equiv (\vec{n}, \vec{u}, \vec{z}_0)$ thus constructed is a right-handed orthonormal basis.

2. The second intermediate basis is denoted by $\boxed{v \equiv (\vec{n}, \vec{v}, \vec{z}_S)}$, where the vector $\vec{v}$ is defined by $\vec{v} = \vec{z}_S \times \vec{n}$. This is a right-handed orthonormal basis.

We use the following terms for the different angles:

- *the precession angle* is $\psi \equiv$ angle $(\vec{x}_0, \vec{n})$ = angle $(\vec{y}_0, \vec{u})$, measured with respect to $\vec{z}_0$.
- *the nutation angle* is $\theta \equiv$ angle $(\vec{z}_0, \vec{z}_S)$ = angle $(\vec{u}, \vec{v})$, measured with respect to $\vec{n}$.
- *the spin angle* is $\varphi \equiv$ angle $(\vec{n}, \vec{x}_S)$ = angle $(\vec{v}, \vec{y}_S)$, measured with respect to $\vec{z}_S$.

The three angles $\boxed{\psi, \theta, \varphi}$ are *the Euler angles* (for the basis $(\vec{x}_S, \vec{y}_S, \vec{z}_S)$ with respect to the basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$). The vector line oriented by $\vec{n}$ is called the *line of nodes*.

We move from the basis $b_0 = (\vec{x}_0, \vec{y}_0, \vec{z}_0)$ to the basis $b_S = (\vec{x}_S, \vec{y}_S, \vec{z}_S)$ through three successive rotations around three different axes (Figure 2.3):

1. We move from basis b_0 to basis $u = (\vec{n}, \vec{u}, \vec{z}_0)$ through the rotation around $\vec{z}_0$, with the rotation angle ψ.
2. We then move from basis u to basis $v = (\vec{n}, \vec{v}, \vec{z}_S)$ through the rotation around $\vec{n}$, with the rotation angle θ.
3. Finally, we move from basis v to basis b_S through a rotation around $\vec{z}_S$, with the rotation angle φ.

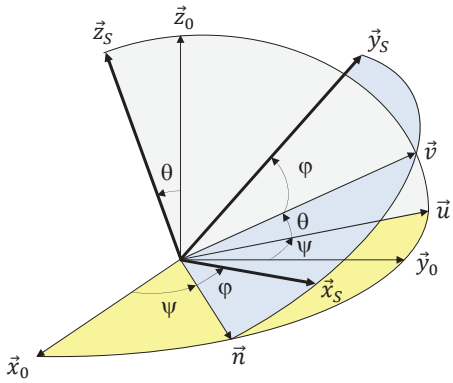


Figure 2.2. *The Euler angles. Global view. For a color version of this figure, see www.iste.co.uk/levan/lagrangian.zip*

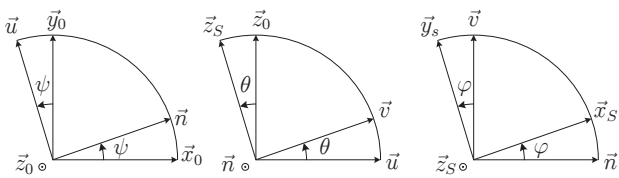


Figure 2.3. *Euler angles. Decomposition in three successive rotations*

Figure 2.3 shows the three rotations and clearly shows that each time we move from one basis to the other, the rotation is carried out around a different common axis.

$$
\begin{array}{lcccc}
\mathrm{b_0}: & & \vec{x}_0 & \vec{y}_0 & \boxed{\begin{matrix}\vec{z}_0\\ \vec{z}_0\end{matrix}} \ \psi/\vec{z}_0 \\
\mathrm{u}: & \theta/\vec{n} & \boxed{\vec{n}} & \vec{u} & \\
\mathrm{v}: & & \boxed{\vec{n}} & \vec{v} & \boxed{\begin{matrix}\vec{z}_S\\ \vec{z}_S\end{matrix}} \ \varphi/\vec{z}_S \\
\mathrm{b_S}: & & \vec{x}_S & \vec{y}_S &
\end{array}
$$

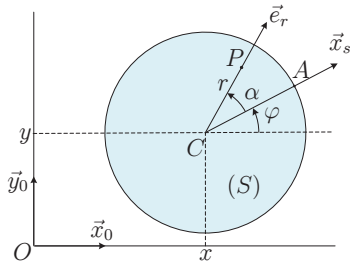
Once the bases b_0, u, v, b_S *have been chosen*, knowing the three Euler angles ψ, θ, φ completely determines the position of basis b_S with respect to b_0. ■

The position of a rectilinear rigid body, which is not reduced to a point, is defined by five parameters: these are the same as those for a non-rectilinear rigid body apart from the rotation around the axis of the rectilinear rigid body.

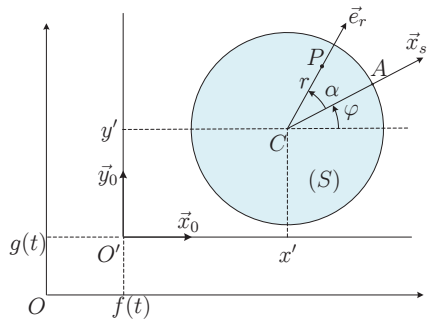
If we consider a plane rigid body moving in the plane $O\vec{x}_0\vec{y}_0$, for example, the position of the rigid body may be defined by the two coordinates of a particle of the rigid body and the rotation angle about the direction orthogonal to the plane of the rigid body.

In certain cases, the position of the rigid body may also explicitly depend on time.

EXAMPLE. Consider a disc S whose center is C and of radius R, moving in a plane endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0)$. Its position may be defined by the two Cartesian coordinates (x, y) of the center C and the angle $\varphi \equiv (\vec{x}_0, \vec{x}_S)$, which is measured with respect to $\vec{z}_0$, where $\vec{x}_S$ is a unit vector attached to the disk S (Figure 2.4(a)).



(a)



(b)

Figure 2.4. *Disc moving in a plane*

Consider a current particle on the disc, whose position P is on a radial line $C\vec{e}_r$ attached to the disc. The distance $r \equiv \|\overrightarrow{CP}\|$ and the angle $\alpha \equiv (\vec{x}_S, \vec{e}_r)$ are constant. The position P can be expressed as a function of the parameters x, y, φ by

$$\overrightarrow{OP} = \overrightarrow{OC} + \overrightarrow{CP} = x\vec{x}_0 + y\vec{y}_0 + r\vec{e}_r(\varphi + \alpha)$$

where the radial vector $\vec{e}_r$ is $\vec{e}_r(\varphi + \alpha) = \cos(\varphi + \alpha)\vec{x}_0 + \sin(\varphi + \alpha)\vec{y}_0$, that is

$$P = P(x, y, \varphi) \qquad [2.4]$$

In certain problems, we may decide to define the position of the disc in another manner by introducing an intermediate coordinate system $(O'; \vec{x}_0, \vec{y}_0)$, which is mobile with respect to the plane, with $\overrightarrow{OO'} = f(t)\vec{x}_0 + g(t)\vec{y}_0$ where $f(t)$, $g(t)$ are known functions of time (Figure 2.4(b)). If x', y' denote the Cartesian coordinates of the center C with respect to this coordinate system, then it is possible to define the position of the disc by x', y', angle φ and time t via $f(t)$, $g(t)$.

The position P of a current particle of the disk can be expressed as a function of the parameters x',y', φ, t through

$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OO'} + \overrightarrow{O'C} + \overrightarrow{CP} = f(t)\vec{x}_0 + g(t)\vec{y}_0 + x'\vec{x}_0 + y'\vec{y}_0 + r\vec{e}_r(\varphi + \alpha) \\ &= (x' + f(t))\,\vec{x}_0 + (y' + g(t))\,\vec{y}_0 + r\vec{e}_r(\varphi + \alpha)\end{aligned}$$

that is:

$$P = P(x', y', \varphi, t) \qquad [2.5]$$

Proceeding in this manner, the position of the disc depends explicitly on the time t. The two points of view [2.4] and [2.5] are equivalent and the choice of one or the other of these is dictated by the context of the problem to be solved. ■

• Generally speaking, it is assumed that the position P, in R_0 and at each instant, of a particle of the rigid body S is a function of N_S variables $q_1^{(S)}, \ldots, q_{N_S}^{(S)}$ and possibly of time t:

$$\begin{aligned} \mathbb{R}^{N_S+1} \supset \Pi \times [0,T] &\rightarrow \mathcal{E} \\ (q_1^{(S)}, \ldots, q_{N_S}^{(S)}, t) &\mapsto \boxed{P = P(q_1^{(S)}, \ldots, q_{N_S}^{(S)}, t)} \end{aligned} \qquad [2.6]$$

where $\Pi \times [0,T]$ is an (open, connected) region of $\mathbb{R}^{N_S+1}$; $[0,T]$ is the time interval of interest; the greatest instant T is often equal to infinity.

Each $(n+1)$-tuple $(q_1^{(S)}, \ldots, q_{N_S}^{(S)}, t)$ corresponds to one position P of the current particle, that is, to one position of the rigid body S.

– The image under the mapping [2.6] is the set of positions of the current particle that can be represented by the mapping. A good mapping [2.6] must be subjective: its image must contain the set of possible positions of the current particle and ideally the entire space $\mathcal{E}$. In other words, any position $P \in \mathcal{E}$ of the current particle relative to R_0 must be the image of *at least* one $(n+1)$-tuple $(q_1^{(S)}, \ldots, q_{N_S}^{(S)}, t) \in \Pi \times [0,T]$.

– On the other hand, the mapping [2.6] is not necessarily injective: a position P may be the image of two distinct $(n+1)$-tuples $(q_1^{(S)}, \ldots, q_{N_S}^{(S)}, t)$.

Hypothesis. For the derivation and integration purposes in the sequel, it is assumed that the mapping [2.6] is of class C^2. [2.7]

We thus exclude impact problems, where velocities are modeled by discontinuous functions and where the mapping [2.6] belongs to class C^0 only.

2.1.3. *Position parameters for a system of rigid bodies*

Let us consider a mechanical system $\mathcal{S}$, made up of a finite number of rigid bodies (some of which may be reduced to particles). Its position $pos_{R_0}(\mathcal{S}, t)$ relative to the reference frame R_0 at an instant t has been defined in [1.16]. This position is known if the mappings [2.6] are known for all the constituent rigid bodies. The set of parameters that makes it possible to determine the position of system $\mathcal{S}$ is the union of all parameters $(q_1^{(S)}, \ldots, q_{N_S}^{(S)})$ of each constituent rigid body, and possibly of t. Denoting this set by $(q_1, \ldots, q_N, t)$, the *a priori* position (relative to the reference frame R_0 and at each instant) of a current particle p of the system, namely $P = pos_{R_0}(p, t)$, is determined by $(q_1, \ldots, q_N)$ and possibly the time t:

$$\boxed{P = P(q_1, \ldots, q_N, t)} \quad \text{or} \quad \boxed{\overrightarrow{OP} = \overrightarrow{OP}(q_1, \ldots, q_N, t)} \quad (O \text{ is the origin of } \mathcal{E}) \qquad [2.8]$$

Definition. The variables $q_1, \ldots, q_N$ are called *the position parameters* for the system. [2.9]

Throughout this book, the expression "position parameter" or simply "parameter" is used instead of the usual expression "generalized coordinate" in the literature.

If the system $\mathcal{S}$ is made up of p_1 non-rectilinear rigid bodies, p_2 rectilinear rigid bodies and p_3 particles, and if each rigid body has its own parameters, then the position of $\mathcal{S}$ in a 3D context depends, *a priori* , on $N = 6p_1 + 5p_2 + 3p_3$ parameters and possibly on time.

REMARK. As the considered bodies are rigid, the description of the position (and thus the motion) of the system is greatly simplified. It is enough to know the value of a *finite* number of

parameters $(q_1, \ldots, q_n, t)$ in order to know the positions of all the particles of the system. In the case of a deformable mechanical system, this is no longer possible; the positions of the particles of the system are defined individually (by a position field or a displacement field), and in an imprecise manner it can be said that the position of the system depends on an infinite number of parameters. ■

2.2. Mechanical joints

There are mechanical problems that involve a single rigid body in space, subject to given at-a-distance forces, that is, the case of a body in free fall which is subjected only to weight or a planet in a celestial mechanics that is subjected to gravitational force. The question of constraints does not arise for such rigid bodies.

However, quite often a rigid body is not alone in space, but is surrounded by other rigid bodies such that it does not move completely freely but is constrained by the presence of neighboring rigid bodies. Physical conditions such as non-interpenetration or no-slip cause restrictions on the relative positions of rigid bodies. They forbid certain *relative motions* between the rigid bodies and permit certain others. We say that the rigid body is subject to *mechanical joints* or simply *joints*.

A joint between two rigid bodies may be realized through direct contact or through intermediate organs. The intermediate organ may be simply a rope connecting rigid bodies or a sophisticated technologic device, such as a ball bearing between a rotary shaft and a bearing. Below are the standard joints in mechanics:

Clamp	Cylindrical joint	Ball-and-cylinder joint
Pivot (or hinged joint)	Spherical joint with pin	Cylinder-and-plane joint
Prismatic joint (or sliding joint)	Spherical joint (or ball joint)	Ball-and-plane joint
Screw joint (or helical joint)	Planar contact	Point contact
Connection by a rope		

From the kinematics point of view, most of the joints that we see in mechanics are *expressed* by mathematical relationships between the position parameters (and their time derivatives) and possibly time itself. These relationships are restrictions on the number of degrees of freedom of the rigid bodies connected. They express the motions allowed by the mechanical joints and are called *constraint equations*. They will be presented in detail in the following section.

From the dynamic point of view, interaction efforts (as defined in [3.1], effort is the generic term that designates force or torque) between two connected rigid bodies must exist at the joint to *ensure* the mechanical connection, i.e. to force the rigid body or rigid bodies to respect the kinematic constraints arising from the joint. These are called *constraint efforts*.

Apart from a few special cases, the constraint efforts are *unknown* just as with certain position parameters. If the mechanical problem is well posed, the problem equations must make it possible to determine the position parameters over time (i.e. the motion of the system) and they must also make it possible to calculate the constraint efforts if desired.

2.3. Constraint equations

Definition. Consider a joint between two rigid bodies in a system or between a rigid body in a system and a body outside the system. This joint is expressed by a certain number of relationships between the position parameters $q_1, \ldots, q_N$, their time derivatives $\dot{q}_1, \ldots, \dot{q}_N$ and possibly time t itself. These relationships are called the *constraint equations*.

One should distinguish between a joint, which is a physical device, and the constraint equations expressing this joint, which are mathematical relationships.

A joint may be rendered by one or more constraint equations. A constraint equation may be an equality or inequality (an inequality is also called an inequation). A *bilateral* joint is expressed by one or more equalities, while a *unilateral* joint is expressed through equalities as well as at least one inequality. In this book, we will focus chiefly on constraint equations that take the form of an equality.

Definition.

1. A constraint equation is *time independent* if it is a relationship only between the position parameters $q_1, \ldots, q_N$ and not time t.
2. A constraint equation is *time dependent* if it is a relationship between the position parameters $q_1, \ldots, q_N$ and time t.
3. A mechanical joint is said to be *time independent* (respectively, *time dependent*) if all the constraint equations expressing this joint are independent of (respectively, dependent on) time.

A relationship that is time independent (respectively, time dependent) is also said to be *scleronomous* (respectively, *rheonomous*), from the Greek *skleros* = hard, *rheo* = flowing and *nomos* = law.

EXAMPLE. Let us return to the example (represented in Figure 2.4) of the disc S moving in a plane and assume that the disc is in contact with the mobile axis $O'\vec{x}_0$. Using the position parameters x', y', φ, t (see Figure 2.4(b)), this contact is expressed by

$$y' = R$$

This is a time-independent constraint equation.

On the other hand, if the position parameters are chosen equal to the coordinates x, y of center C, defined with respect to the fixed coordinate system $(O; \vec{x}_0, \vec{y}_0)$ (see Figure 2.4(a)), and the rotation angle φ, then the contact considered is expressed by

$$y = R + g(t)$$

which is, here, a constraint equation that is time dependent. This example shows how the same mechanical joint may be expressed by equations that are time dependent or time independent depending on the choice of the position parameters. ■

The constraint equations and the constraint efforts exist in a concomitant manner for each given joint. Each constraint equation *induces* or *provokes* certain constraint efforts and, conversely, each constraint force *results from* or *ensures* certain constraint equations.

EXAMPLE. In the previous example, the contact between the disc and the mobile axis was expressed by the equation $y' = R$. If the contact takes place with friction, it induces two contact forces – one that is normal and one that is tangential. ■

Definitions. [2.10]

1. A constraint equation is said to be *holonomic* if it is a relationship between the position parameters and possibly time, without involving the time derivatives of the position parameters. It is, thus, of the form

$$\boxed{f(q_1, \ldots, q_N, t) = 0} \qquad (\text{or } \geq 0 \text{ or } \leq 0 \text{ if there is an inequality})$$

2. A constraint equation is said to be *non-holonomic* if it is a relationship between the position parameters and possibly time, together with the time derivatives of the position parameters. It thus takes the form

$$\boxed{f(q_1, \ldots, q_N, \dot{q}_1, \ldots, \dot{q}_N, t) = 0} \qquad (\text{or } \geq 0 \text{ or } \leq 0 \text{ if there is an inequality})$$

3. A constraint equation is said to be *semi-holonomic* (or integrable) if it is a non-holonomic relationship that can be integrated with respect to time in order to reduce it to the form

$$\boxed{f(q_1, \ldots, q_N, t) = C}$$

where C is a constant of integration that depends on the initial conditions.

The term "holonomic" comes from the Greek *holos* = whole and *nomos* = law.

Definition. [2.11]

A constraint equation that is said to be *solved* is a particular holonomic equation that gives a certain parameter as a function of some others (not necessarily all other parameters). For example:

$$\boxed{q_{n+1} = \chi(q_1, \ldots, q_n, t)} \qquad \text{where } n < N$$

The parameter expressed as a function of the others is called a *solved* parameter.

In what follows, we will only consider the non-holonomic constraint equations, which take the following *differential form*:

$$\boxed{\sum_{k=1}^{N} \alpha_k(q_1, \ldots, q_N, t)\dot{q}_k + \beta(q_1, \ldots, q_N, t) = 0} \qquad [2.12]$$

Throughout the following, it is assumed that all constraint equations considered are *independent*. In other words, no constraint equation can be obtained by transforming other constraint equations (transforming by linear combination, derivation, integration, etc.).

With regard to the mathematical smoothness, we assume the following conditions:

Assumptions. [2.13]

– The functions $f(q_1, \ldots, q_N, t)$ in [2.10], especially the χ functions in [2.11], are of class C^2 over their domain.

– The functions α_k and β in [2.12] are of class C^1 over their domains.

EXAMPLES.

1. Consider a particle with position P, moving with respect to the reference frame R_0 endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$. For the position parameters of the particle, we can choose between the Cartesian coordinates x, y, z of the point P and the spherical coordinates r, φ, θ defined with respect to the center O.

   Assume that the particle is connected to the fixed point O through an inextensible wire of length ℓ. The mechanical joint is expressed by

$$x^2 + y^2 + z^2 = \ell^2 \qquad \text{or} \qquad r = \ell$$

The two constraint equations are holonomic. The second equation is solved contrary to the first one.

This example shows that depending on the choice of position parameters, the equations expressing the same mechanical joint may have more or less simple forms.

2. Let us return to the example (represented in Figure 2.4(a)) of a disc moving in a plane, where the position parameters are x, y, φ (see relation [2.4]).

   Assume that the disc is in no-slip contact with the axis $O\vec{x}_0$, at the point denoted by I. The contact at I is expressed by

$$y = R \qquad [2.14]$$

   This is a solved constraint equation. The no-slip contact condition at I can be written as

$$\vec{0} = \vec{V}_{0S}(I) = \vec{V}_{0S}(C) + \vec{\Omega}_{0S} \times \overrightarrow{CI} = \dot{x}\vec{x}_0 + \dot{y}\vec{y}_0 + \dot{\varphi}\vec{z}_0 \times (-R\vec{y}_0) = (\dot{x} + R\dot{\varphi})\vec{x}_0 + \dot{y}\vec{y}_0$$

   that is

$$\begin{aligned} \dot{x} + R\dot{\varphi} &= 0 \\ \dot{y} &= 0 \end{aligned}$$

   The last relationship is a consequence of [2.14]. The first relationship is a semi-holonomic constraint equation as it may be integrated with respect to time to arrive at the form $x + R\varphi = C$, where the constant of integration $C = x_0 + R\varphi_0$ depends on the initial conditions.

3. Consider a disc S with center C and radius a, moving in the reference frame R_0 endowed with the orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$. For the position parameters of the disc, we choose the coordinates x, y, z of the center C and the Euler angles ψ, θ, φ defined in [2.3] (Figure 2.5).

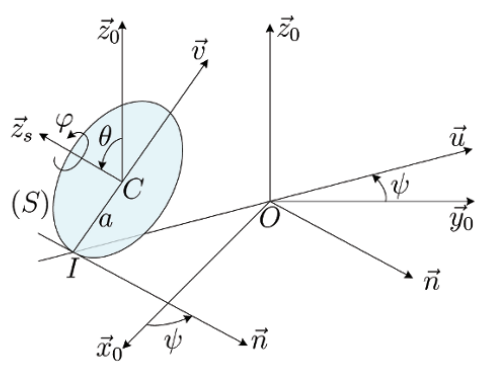


Figure 2.5. *Disc in contact with a plane*

   Assume that the disc is in no-slip contact with the plane $O\vec{x}_0\vec{y}_0$ at the point I. The contact at I is expressed by

$$z = a \sin\theta \qquad [2.15]$$

   This is a holonomic, even solved, constraint equation. To express the no-slip contact condition at I, let us use the vectors of the intermediate basis $v \equiv (\vec{n}, \vec{v}, \vec{z}_S)$ represented in

Figure 2.5. The vector $\vec{z}_S$ is orthogonal and attached to the disc S. The no-slip contact condition is written as

$$\vec{0} = \vec{V}_{0S}(I) = \vec{V}_{0S}(C) + \vec{\Omega}_{0S} \times \overrightarrow{CI} = \dot{x}\vec{x}_0 + \dot{y}\vec{y}_0 + \dot{z}\vec{z}_0 + (\dot{\psi}\vec{z}_0 + \dot{\theta}\vec{n} + \dot{\varphi}\vec{z}_S) \times (-a\vec{v})$$

That is, projecting on to the basis $u \equiv (\vec{n}, \vec{u}, \vec{z}_0)$:

$$\begin{array}{lrcl} /\vec{n}: & \dot{x}\cos\psi + \dot{y}\sin\psi + a(\dot{\varphi} + \dot{\psi}\cos\theta) & = & 0 \\ /\vec{u}: & -\dot{x}\sin\psi + \dot{y}\cos\psi + a\dot{\theta}\sin\theta & = & 0 \\ /\vec{z}_0: & \dot{z} - a\dot{\theta}\cos\theta & = & 0 \end{array}$$

The last relationship is a consequence of [2.15]. The first two relationships are non-holonomic constraint equations of the differential form [2.12]. ■

2.4. Parameterization

Let us consider a mechanical system $\mathcal{S}$ whose *a priori* position in R_0 is described by N position parameters, $q_1, \ldots, q_N$ (see [2.9]) and possibly by time. We assume that there exist mechanical joints in the system, which are expressed by a certain number of constraint equations.

We will choose some *solved* (see definition [2.11]) constraint equations to *eliminate* certain parameters in favor of others. There is no difficulty in the elimination of parameters as we exploit resolved equations. Furthermore, it is clear that we cannot use non-holonomic or semi-holomonic constraint equations (see definition [2.10]). Using an appropriate ordering of parameters, we can always assume that we preserve n first parameters $q_1, \ldots, q_n$ and leave out others, where n is a chosen number ($n \leq N$ and the lower bound for n depends on the number of solved constraint equations available).

Definitions. [2.16]

1. In analytical mechanics, the position parameters, *a priori*, $q_1, \ldots, q_N$ of the system and possibly the time t are called *primitive parameters* to distinguish them from the retained parameters defined below.

2. The *solved* constraint equations used to eliminate $q_{n+1}, \ldots, q_N$ are called *primitive constraint equations*. They are of the form

$$\boxed{\begin{array}{lcl} q_{n+1} & = & \chi_{n+1}(q_1, \ldots, q_n, t) \\ & \vdots & \\ q_N & = & \chi_N(q_1, \ldots, q_n, t) \end{array}} \qquad [2.17]$$

3. *The retained parameters* are, by definition, the parameters that have been preserved: $\boxed{q_1, \ldots, q_n, t}$ or, more briefly, $\boxed{q, t}$ with $\boxed{q \equiv (q_1, \ldots, q_n)}$.

   We can, thus, express all kinematic quantities that appear in what follows as functions of (q, t) solely. In particular, the position P, in R_0 and at each instant, of a current particle of the system is a function of (q, t):

$$\boxed{P = P(q, t)}$$

4. The other constraint equations, that is, those which were not used to eliminate $q_{n+1}, \ldots, q_N$, are called the *complementary constraint equations*.

   Using [2.17], we can express the complementary constraint equations in terms of the parameters (q, t).

A parameterization of the system consists of the four points listed above.

The parameterization defined in the above sense does not exist in Newtonian mechanics. On the other hand, it is indispensable in analytical mechanics and is the first task to be carried out in any problem. In order to establish a parameterization, the physicist must:

– choose *the primitive parameters* of the system studied. In practice, these parameters appear naturally when we address the mechanical problem. Other position parameters may be discarded from the start, if they are not useful;

– choose *the primitive constraint equations*. This is a subjective choice guided by the objective specified by the physicist. Indeed:

  • it will be seen that when a given constraint equation is classified as primitive, the constraint efforts associated with this equation do not appear in the governing equations and thus become inaccessible;

  • if we wish to obtain a particular constraint effort, the constraint equations corresponding to this effort must be classified as complementary.

Note that a non-holonomic constraint equation cannot be used to express a parameter as a function of other parameters. It cannot, thus, be chosen as a primitive constraint equation.

The retained parameters and the complementary constraint equations result from the above choice. It should once again be emphasized that the choice of categorizing one constraint equation as primitive and another as complementary, i.e. the choice of the retained parameters, is the responsibility of the physicist and depends on the chosen objective.

The expression for the virtual velocity fields (VVF), the Lagrange's equations that we will obtain and, therefore the mechanical information that we can derive from these equations, are all dependent on the chosen parameterization.

For a given mechanical system, there are often several possible parameterizations. We can distinguish between the three following types of parameterization.

Definitions. [2.18]

– The *total* parameterization consists of taking *all* the primitive parameters as the parameters to be retained ($n = N$). In other words, there is no primitive constraint equation and all the constraint equations are written as complementary equations.

– The parameterization is said to be *reduced* if the number of retained parameters, n, is strictly smaller than N. In this case, a certain number of (solved) constraint equations are chosen to be primitive and then used to eliminate certain primitive parameters.

– The parameterization is said to be *independent* if there exists no complementary constraint equation. This is a minimal reduced parameterization, in the sense that all constraint equations are classified as primitive.

The reduced parameterization is an intermediate category between two extreme cases: total parameterization and independent parameterization. The typical parameterization encountered in this book is, therefore, the following reduced parameterization:

Parameterization. [2.19]

It is assumed that the mechanical system $\mathcal{S}$ is given in the following parameterization:

1. Primitive parameters: $q_1, \ldots, q_N, t$.

2. Primitive constraint equations:

$$\boxed{\begin{aligned} q_{n+1} &= \chi_{n+1}(q_1, \ldots, q_n, t) \\ &\vdots \\ q_N &= \chi_N(q_1, \ldots, q_n, t) \end{aligned}} \qquad [2.20]$$

3. Retained parameters: $q_1, \ldots, q_n, t$, or in abridged form (q, t).

   The position P, in R_0 and at the instant t, of a current particle of the system is

$$\boxed{P = P(q, t)} \qquad [2.21]$$

4. Complementary constraint equations: they are specified depending on the problem studied.

The mapping $(q, t) \mapsto P(q, t)$ is of class C^2. [2.22]
Indeed:

– according to hypothesis [2.7], the mapping $(q_1, \ldots, q_N, t) \mapsto \overrightarrow{O_1P}(q_1, \ldots, q_N, t)$ belongs to class C^2,

– moreover, according to hypothesis [2.13], the functions $\chi_{n+1}, \ldots, \chi_N$ in [2.20] are of class C^2 over their domain.

2.5. Dependence of the rotation tensor of the reference frame on the retained parameters

In the following, we will bring in a reference frame R_1 that is different from the common reference frame R_0. The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 often depends on t, but not on the position parameters q. In certain cases, however, it turns out that the chosen parameterization implies that $\bar{\bar{Q}}_{01}$ also depends on q. For a general case, we will thus write *a priori*

$$\boxed{\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{01}(q, t)} \qquad [2.23]$$

With $q = q(t)$, the tensor $\bar{\bar{Q}}_{01}(q, t)$, for example, is a *composite* function of time, which is denoted by $\bar{\bar{Q}}_{01}(t)$ in Chapter 1. In Chapter 1, the notation $\bar{\bar{Q}}_{01}(t)$ sufficed because only the dependence with respect to time was important. From this point onwards, however, we must keep in mind the dependence [2.23], *a priori*, even though we will not systematically write the arguments q, t explicitly.

EXAMPLE. Consider a 2D problem where two reference frames are involved: the common reference frame R_0 endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0)$ and another reference frame, R_1, endowed with an orthonormal coordinate system $(O_1; \vec{x}_1, \vec{y}_1)$. Let X, Y denote the Cartesian coordinates of the point O_1 relative to the coordinate system $(O; \vec{x}_0, \vec{y}_0)$, and let φ be the angle $(\vec{x}_0, \vec{x}_1)$ measured around $\vec{z}_0 \equiv \vec{x}_0 \times \vec{y}_0$ and assume that $\varphi = \omega t$ where ω is a given constant (Figure 2.6).

The tensor $\bar{\bar{Q}}_{01}$ is the rotation of angle φ around $\vec{z}_0$, which rotates the vectors $\vec{x}_0, \vec{y}_0$ into the vectors $\vec{x}_1, \vec{y}_1$, respectively.

Consider a rod S whose position in R_0 is AB. The coordinates of point A relative to the coordinate system $(O_1; \vec{x}_1, \vec{y}_1)$ are denoted byx, y, and the angle $(\vec{x}_1, \overrightarrow{AB})$ measured around $\vec{z}_0$

is θ. We will parameterize the position of the rod relative to R_0 using two different parameterizations.

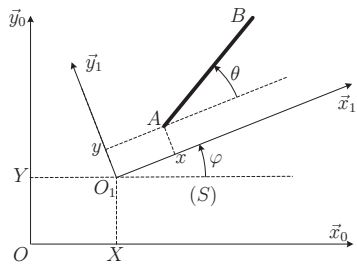


Figure 2.6. *Example of parameters*

1. First parameterization:

   (a) Primitive parameters: we choose to define the position of the rod S in R_0 using the coordinates X, Y of point O_1, the coordinates x, y of point A and the angles φ, θ.

   (b) Primitive constraint equations: it is decided that no constraint equation is classified as primitive.

   (c) The retained parameters are, therefore: $q = (X, Y, x, y, \varphi, \theta)$.

   The position P, in R_0 and at the instant t, of any particle p of the rod is $P = P(X, Y, x, y, \varphi, \theta)$.

   (d) Complementary constraint equations: the constraint equation $\varphi = \omega t$ is thus found to be complementary.

   Using this parameterization, the rotation tensor $\bar{\bar{Q}}_{01}$ depends on the parameters q via φ.

2. Second parameterization: we will now consider a different parameterization where the constraint equation $\varphi = \omega t$ is this time classified as primitive and not complementary.

   (a) Primitive parameters: these are the same as in the first parameterization.

   (b) Primitive constraint equations: we decide to classify the constraint equation $\varphi = \omega t$ as primitive.

   (c) The retained parameters are, therefore, $q = (X, Y, x, y, \theta)$ and t; the dependence with respect to t occurs via $\varphi = \omega t$.

   The position P, in R_0 and at the instant t, of the current particle p is $P = P(X, Y, x, y, \theta, t)$.

   (d) Complementary constraint equation: none.

   Using this parameterization, the rotation tensor $\bar{\bar{Q}}_{01}$ does not depend on q. ■

From this example, the following general observations can be drawn for a system that is made up of one or more rigid bodies, whose retained parameters are (q, t) as follows:

1. If the reference frame R_1 is defined by a rigid body (in the system studied) whose position in R_0 depends on q, the rotation tensor $\bar{\bar{Q}}_{01}$ will obviously depend on q.

2. However, even if the reference frame R_1 is not defined by a rigid body belonging to the system, the rotation tensor $\bar{\bar{Q}}_{01}$ may depend on q. In the previous example, R_1 is not defined by the rod and yet it may still depend on q.

For the rest of the chapter:

1. up to section 2.10, we will study the general case when $\bar{\bar{Q}}_{01}$ depends on (q, t), and we will then derive the case when $\bar{\bar{Q}}_{01}$ does not depend on q as a particular case;

2. from section 2.10 onwards, we will study only the case when $\bar{\bar{Q}}_{01}$ does not depend on q.

The case when $\bar{\bar{Q}}_{01}$ depends on (q, t) leads to more complex expressions, but it is indispensable in certain situations, as will be seen in the following chapters.

• One of the first uses of [2.23] is to derive, from [2.21], the position $P^{(1)} = pos_{R_1}(p, t)$ of the current particle in any reference frame R_1:

$$\begin{aligned}\overrightarrow{OP^{(1)}} &= \bar{\bar{Q}}_{10}(q,t).\overrightarrow{O_1P} && \text{according to [1.26], where } O_1 \text{ is the origin of}\\ &= \bar{\bar{Q}}_{10}(q,t).(\overrightarrow{OP} - \overrightarrow{OO_1}) && \text{the coordinate system of } R_1\\ &= \bar{\bar{Q}}_{10}(q,t).\overrightarrow{OP}(q,t) - \bar{\bar{Q}}_{10}(q,t).\overrightarrow{OO_1}\end{aligned}$$

This signifies that the position $P^{(1)}$ of p in R_1 is also a function of q, t:

$$\boxed{\overrightarrow{OP^{(1)}} = \overrightarrow{OP^{(1)}}(q,t)} \qquad [2.24]$$

Consequently, the position of the mechanical system $\mathcal{S}$, in any reference frame and at any instant, is determined by the parameters q and possibly the time t.

2.6. Velocity of a particle

The objectives considered here and for the longer term are the following:

– We go back to the velocity [1.47] of a particle p and we will give it an explicit expression by taking into account the fact that here the position P of the particle in R_0 is given by [2.21].

– The obtained expression will serve as the model that will be used in Chapter 4 to adopt the definition of the virtual velocity (VV) of a particle with respect to R_1.

– In Chapter 5, this VV will in turn be used to define the virtual power (VP) of efforts and the VP of the quantities of acceleration.

In Chapter 5, it will be seen that the relationships obtained for the VP involve the VV with respect to a reference frame R_1 such that $\bar{\bar{Q}}_{01}$ may or may not depend on the retained parameters q. The typical example is that of expression [5.14] for the VP of interefforts between two rigid bodies, where there appears the VV relative to the reference frame defined by one of the two rigid bodies and where the rotation tensor of this reference frame depends on q. Thus, to prepare to study the VP in Chapter 5, we will now investigate the velocity in two steps:

1. we will begin by considering the general case when $\bar{\bar{Q}}_{01}$ may or may not depend on q;
2. from this we will then derive the results for the particular case when $\bar{\bar{Q}}_{01}$ does not depend on q.

Theorem. Let p be a particle whose position $P(q,t)$ in R_0 is given by [2.21]. Its velocity, with respect to the reference frame R_1 and at the instant t, is given as

$$\boxed{\vec{V}_{R_1}(p,t) = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}}, \qquad [2.25]$$

where $\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{01}(q,t)$ and the partial derivative $\dfrac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}$ is defined in [1.41]. The derivatives $\dfrac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)$ are the standard derivatives of the vector $\bar{\bar{Q}}_{10}.\overrightarrow{O_1P} \in E$ with respect to q_i. They are not the derivatives with respect to R_1.

PROOF. According to [1.47], we have $\vec{V}_{R_1}(p,t) = \bar{\bar{Q}}_{01}.\dfrac{d}{dt}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)$. Taking into account [2.21], the final derivative is written as

$$\frac{d}{dt} = \sum_{i=1}^{n} \frac{\partial}{\partial q_i}\dot{q}_i + \frac{\partial}{\partial t} \qquad \blacksquare$$

Let us introduce the following hypothesis which is often – but not always – satisfied:

Hypothesis. The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 depends only on time and *not* on q. [2.26]

If this hypothesis is satisfied, one may simplify relationship [2.25] in a straightforward manner:

Theorem. Let p be a particle whose position $P(q,t)$ in R_0 is given by [2.21]. Using hypothesis [2.26], the velocity of the particle p, with respect to the reference frame R_1 and at the instant t, is

$$\boxed{\vec{V}_{R_1}(p,t) = \sum_{i=1}^{n} \frac{\partial\overrightarrow{O_1P}}{\partial q_i}\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}} \qquad [2.27]$$

The derivatives $\dfrac{\partial\overrightarrow{O_1P}}{\partial q_i}$ are the ordinary derivatives of $\overrightarrow{O_1P} \in E$ with respect to q_i. They are not the derivatives with respect to R_1.

Relationship [2.27] is simpler than [2.25], however *it can only be used under hypothesis* [2.26].

EXAMPLE AND COUNTER-EXAMPLE. Let us return to the example in section 2.5 and calculate the velocity – with respect to R_1 (respectively, R_0) – of the particle a with the position A, using the two different parameterizations considered in section 2.5.

1. First parameterization.
   (a) Let us calculate the velocity of the particle a with respect to R_1 using the first parameterization. We can directly apply [2.25] or (which amounts to the same thing)

start from definition [1.47]:

$$\vec{V}_{R_1}(a,t) \equiv \frac{d_{R_1}\overrightarrow{O_1A}}{dt} \equiv \bar{\bar{Q}}_{01}.\frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}\right) \qquad [2.28]$$

where $\overrightarrow{O_1A} = x\vec{x}_1(\varphi) + y\vec{y}_1(\varphi)$. Straightforward calculations lead successively to

$$\begin{aligned}
\bar{\bar{Q}}_{10}.\overrightarrow{O_1A} &= x\vec{e}_1 + y\vec{e}_2 \qquad \text{recall that } (\vec{e}_1,\vec{e}_2) \text{ is the basis of } E, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{and [1.30]: } (\vec{x}_0,\vec{y}_0) = (\vec{e}_1,\vec{e}_2) \\
\Rightarrow \frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}\right) &= \dot{x}\vec{e}_1 + \dot{y}\vec{e}_2 \\
\Rightarrow \bar{\bar{Q}}_{01}.\frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}\right) &= \dot{x}\vec{x}_1 + \dot{y}\vec{y}_1
\end{aligned}$$

or

$$\vec{V}_{R_1}(a,t) = \dot{x}\vec{x}_1 + \dot{y}\vec{y}_1 \qquad [2.29]$$

As the rotation tensor $\bar{\bar{Q}}_{01}$ here depends on φ, hypothesis [2.26] is not satisfied and we *cannot* use [2.27]. Indeed, relationship [2.27] would give us

$$\begin{aligned}
\vec{V}_{R_1}(a,t) &= \sum_{i=1}^{n}\frac{\partial\overrightarrow{O_1A}}{\partial q_i}\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1A}}{\partial t} \\
&= \underbrace{\frac{\partial\overrightarrow{O_1A}}{\partial x}}_{\vec{x}_1}\dot{x} + \underbrace{\frac{\partial\overrightarrow{O_1A}}{\partial y}}_{\vec{y}_1}\dot{y} + \underbrace{\frac{\partial\overrightarrow{O_1A}}{\partial \varphi}}_{x\vec{y}_1 - y\vec{x}_1}\dot{\varphi} + \underbrace{\frac{\partial_{R_1}\overrightarrow{O_1A}}{\partial t}}_{\vec{0}} \\
&= (\dot{x} - y\dot{\varphi})\vec{x}_1 + (\dot{y} + x\dot{\varphi})\vec{y}_1 \qquad \text{which is incorrect.}
\end{aligned}$$

(b) For the purposes of comparison, let us calculate the velocity with respect to R_0:

$$\vec{V}_{R_0}(a,t) \equiv \frac{d_{R_0}\overrightarrow{OA}}{dt} \equiv \bar{\bar{Q}}_{00}.\frac{d}{dt}\left(\bar{\bar{Q}}_{00}.\overrightarrow{OA}\right) \underset{\bar{\bar{Q}}_{00}=\bar{\bar{I}}}{=} \frac{d\overrightarrow{OA}}{dt}, \qquad [2.30]$$

where $\overrightarrow{OA} = X\vec{e}_1 + Y\vec{e}_2 + x\vec{x}_1(\varphi) + y\vec{y}_1(\varphi)$. We obtain

$$\vec{V}_{R_0}(a,t) = \dot{X}\vec{e}_1 + \dot{Y}\vec{e}_2 + (\dot{x} - y\dot{\varphi})\vec{x}_1 + (\dot{y} + x\dot{\varphi})\vec{y}_1 \qquad [2.31]$$

2. Second parameterization.

(a) Let us now use the second parameterization from the example in section 2.5. Using relationship [2.28], this time with $\overrightarrow{O_1A} = x\vec{x}_1(t) + y\vec{y}_1(t)$, and the same calculation as in the first parameterization, we obtain

$$\vec{V}_{R_1}(a,t) = \dot{x}\vec{x}_1 + \dot{y}\vec{y}_1 \qquad [2.32]$$

which is identical to [2.29] on the condition, of course, that in [2.29] we make $\vec{x}_1 = \vec{x}_1(\varphi) = \vec{x}_1(\omega t)$, $\vec{y}_1 = \vec{y}_1(\varphi) = \vec{y}_1(\omega t)$, and in [2.32] we make $\vec{x}_1 = \vec{x}_1(t)$, $\vec{y}_1 = \vec{y}_1(t)$.

Moreover, given that here the rotation tensor $\bar{\bar{Q}}_{01}$ does not depend on q, we can use [2.27]:

$$\begin{aligned}
\vec{V}_{R_1}(a,t) &= \sum_{i=1}^{n}\frac{\partial\overrightarrow{O_1A}}{\partial q_i}\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1A}}{\partial t} \\
&= \underbrace{\frac{\partial\overrightarrow{O_1A}}{\partial x}}_{\vec{x}_1}\dot{x} + \underbrace{\frac{\partial\overrightarrow{O_1A}}{\partial y}}_{\vec{y}_1}\dot{y} + \bar{\bar{Q}}_{01}.\underbrace{\frac{\partial}{\partial t}\Big(\underbrace{\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}}_{x\vec{e}_1+y\vec{e}_2}\Big)}_{\vec{0}}
\end{aligned}$$

We once again arrive at [2.32].

(b) The calculation of the velocity with respect to R_0 is carried out in the same manner. Using relationship [2.30], this time with $\overrightarrow{OA} = X\vec{e}_1 + Y\vec{e}_2 + x\vec{x}_1(t) + y\vec{y}_1(t)$, leads to

$$\vec{V}_{R_0}(a,t) = \dot{X}\vec{e}_1 + \dot{Y}\vec{e}_2 + (\dot{x} - \omega y)\vec{x}_1 + (\dot{y} + \omega x)\vec{y}_1$$

which is identical to [2.31], knowing that $\varphi = \omega t$. ■

Definition. In general, the vectors $\dfrac{\partial \overrightarrow{O_1P}}{\partial q_i}$, $i \in [1,n]$, which appear in [2.27], form a linearly independent set of vectors (except, perhaps, at some special points). In this case, the parameterization is said to be *regular* and the linearly independent set of vectors is called the *local basis at* P.

In general, the local basis at P depends on (q,t) and, therefore, on the position P. In the case when P depends on the Cartesian coordinates $P = P(x,y,z)$, the local basis is merely the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ of E and is independent of the position P.

• Below is a hypothesis that is slightly stronger than [2.26] and that is often but not always satisfied:

Hypothesis. The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and the point O_1 fixed in R_1 does *not* depend on q. [2.33]

This hypothesis makes it possible to simplify [2.27] a little. Indeed, let us write $\overrightarrow{O_1P} = \overrightarrow{O'P} - \overrightarrow{O'O_1}$, where O' is any point other than O_1 and independent of q. We thus have $\dfrac{\partial \overrightarrow{O_1P}}{\partial q_i} = \dfrac{\partial \overrightarrow{O'P}}{\partial q_i} - \dfrac{\partial \overrightarrow{O'O_1}}{\partial q_i}$ where $\dfrac{\partial \overrightarrow{O'O_1}}{\partial q_i} = \vec{0}$ as the points O_1, O' do not depend on q. We can thus rewrite [2.27] in a simpler form

$$\boxed{\vec{V}_{R_1}(p,t) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{P}}{\partial q_i}\dot{q}_i + \frac{\partial_{R_1} \overrightarrow{O_1P}}{\partial t}} \qquad [2.34]$$

where $\dfrac{\partial \overrightarrow{P}}{\partial q_i}$ denotes $\dfrac{\partial \overrightarrow{O'P}}{\partial q_i}$, O' being any point that is independent of q.

2.7. Angular velocity

Theorem and definition. Consider a reference frame R_2 different from R_1 and assume that the position of the rigid body $S(R_2)$ defined by R_2 depends on (q,t) (which is often true since, in practical problems, R_2 is defined by a rigid body belonging to the mechanical system studied).

The angular velocity vector $\vec{\Omega}_{12}$ of the reference frame R_2 with respect to the reference frame R_1 is written as the sum of a linear form of $\dot{q}_i$ and a constant term with respect to $\dot{q}_i$:

$$\boxed{\vec{\Omega}_{12} = \sum_{i=1}^{n} \vec{\omega}_{12}^{\,i}\dot{q}_i + \vec{\omega}_{12}^{\,t}} \qquad [2.35]$$

where $\vec{\omega}_{12}^{\,i}$ and $\vec{\omega}_{12}^{\,t}$, called the *partial angular velocities of* R_2 *with respect to* R_1, are defined as

$$\forall i \in [1,n], \boxed{\vec{\omega}_{12}^{\,i} \equiv \frac{1}{2}\sum_{j=1}^{3} \vec{b}_j \times \left(\bar{\bar{Q}}_{01}\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j)\right)} \quad \text{and} \quad \boxed{\vec{\omega}_{12}^{\,t} \equiv \frac{1}{2}\sum_{j=1}^{3} \vec{b}_j \times \frac{\partial_{R_1}\vec{b}_j}{\partial t}} \qquad [2.36]$$

where $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ is an orthonormal basis *fixed in* R_2 (refer again to Figure 1.8) and where we recall that the partial derivative $\dfrac{\partial_{R_1}\vec{b}_j}{\partial t}$ is defined in [1.41].

Note that the physical unit of $\vec{\omega}_{12}^t$ is rad/s, while the unit for $\vec{\omega}_{12}^i$ is rad/s divided by the unit of $\dot{q}_i$.

PROOF. Let us recall expression [1.48] for the angular velocity vector of a reference frame R_2 with respect to another reference frame R_1:

$$\vec{\Omega}_{12} \equiv \vec{\Omega}_{R_1R_2} \equiv \frac{1}{2}\sum_{j=1}^{3} \vec{b}_j \times \frac{d_1\vec{b}_j}{dt}$$

As the position of the rigid body $S(R_2)$ defined by R_2 is given by [2.21], we have

$$\frac{d_1\vec{b}_j}{dt} \equiv \bar{\bar{Q}}_{01}.\frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right) = \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right)\dot{q}_i + \frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right)\right) \qquad \blacksquare$$

Relationship [2.35] enables one to interpret $\vec{\omega}_{12}^i$ and $\vec{\omega}_{12}^t$ as the partial angular velocities of $S(R_2)$: $\vec{\omega}_{12}^i$ is the angular velocity when all position parameters other than q_i as well as time t are kept constant, while $\vec{\omega}_{12}^t$ is the angular velocity when all position parameters are kept constant and only t varies.

Let $\mathcal{S}$ be a system of rigid bodies whose position is defined by the parameters $q_1, \ldots, q_n, t$ (see [2.21]) and consider a rigid body in the system, named S_2, that defines a reference frame R_2. It may be that the position of S_2 in R_1 – more precisely, the vectors $\bar{\bar{Q}}_{10}.\vec{b}_j$ – does not depend on a particular parameter q_i or on t. In this case, the vector $\vec{\omega}_{12}^i$ or the vector $\vec{\omega}_{12}^t$ are zero.

2.8. Velocities in a rigid body

Let us return to section 1.9 concerning the velocities in a rigid body and add a few more results, considering the fact that here the position $P(q,t)$ in R_1 of a current particle in the rigid body is given by [2.21].

Lemma. Consider a reference frame R_1, *a rigid body* S defining a reference frame R_S, and an orthonormal basis $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ *fixed in* R_S. We have the following equalities that hold $\forall t$, $\forall q$ (however, in order to simplify the writing, the arguments q, t are left out):

$$\begin{aligned} &\forall i \in [1,n], \forall j \in [1,3], \quad \boxed{\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right) = \vec{\omega}_{1S}^i \times \vec{b}_j} \\ &\forall j \in [1,3], \quad \boxed{\frac{\partial_{R_1}\vec{b}_j}{\partial t} \underset{[1.41]}{\equiv} \bar{\bar{Q}}_{01}.\frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right) = \vec{\omega}_{1S}^t \times \vec{b}_j} \end{aligned} \qquad [2.37]$$

where $\vec{\omega}_{1S}^i$, $\vec{\omega}_{1S}^t$ are the partial angular velocities of S with respect to R_1, defined in [2.35].

- If we add hypothesis [2.26], then expression $[2.37]_1$ immediately simplifies to

$$\forall i \in [1,n], \forall j \in [1,3], \quad \boxed{\frac{\partial \vec{b}_j}{\partial q_i} = \vec{\omega}_{1S}^i \times \vec{b}_j} \qquad [2.38]$$

where $\vec{\omega}_{1S}^i$ is derived from [2.36]: $\vec{\omega}_{1S}^i \equiv \dfrac{1}{2}\displaystyle\sum_{j=1}^{3}\vec{b}_j \times \frac{\partial \vec{b}_j}{\partial q_i}$.

FIRST PROOF. The demonstration will be carried out with the fixed indices i, j, $i \in [1, n]$, $j \in [1, 3]$. Upon applying [2.36], we have

$$\begin{aligned}
\vec{\omega}_{1S}^{i} \times \vec{b}_j &= \left[\frac{1}{2}\sum_{k=1}^{3} \vec{b}_k \times \left(\bar{\bar{Q}}_{01}\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_k)\right)\right] \times \vec{b}_j \\
&= \frac{1}{2}\bar{\bar{Q}}_{01}\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j) - \frac{1}{2}\sum_{k=1}^{3}\left[\vec{b}_j.\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_k)\right]\vec{b}_k \\
&\qquad \text{using } (\vec{a} \times \vec{b}) \times \vec{c} = (\vec{c}.\vec{a})\vec{b} - (\vec{c}.\vec{b})\vec{a}
\end{aligned}$$

where

$$\begin{aligned}
\vec{b}_j.\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_k) &= (\bar{\bar{Q}}_{10}.\vec{b}_j).\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_k) \qquad \text{because } \vec{b}_j.\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{10}.\vec{b}_j \\
&= \frac{\partial}{\partial q_i}\underbrace{\left[(\bar{\bar{Q}}_{10}\vec{b}_j).(\bar{\bar{Q}}_{10}\vec{b}_k)\right]}_{=\vec{b}_j.\bar{\bar{Q}}_{01}.\bar{\bar{Q}}_{10}.\vec{b}_k=\vec{b}_j.\vec{b}_k} - (\bar{\bar{Q}}_{10}.\vec{b}_k).\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j) \\
&= \vec{0} - \vec{b}_k.\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j)
\end{aligned}$$

Hence

$$\begin{aligned}
\vec{\omega}_{1S}^{i} \times \vec{b}_j &= \frac{1}{2}\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j) + \frac{1}{2}\sum_{k=1}^{3}\left[\vec{b}_k.\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j)\right]\vec{b}_k \\
&= \frac{1}{2}\bar{\bar{Q}}_{01}\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j) + \frac{1}{2}\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{10}\vec{b}_j) \qquad \text{which is } [2.37]_1.
\end{aligned}$$

Relationship $[2.37]_2$ can be proved in a similar manner by replacing $\partial/\partial q_i$ with $\partial/\partial t$. ■

SECOND PROOF. The proof will be carried out with a fixed index $j \in [1, 3]$. As the vector $\vec{b}_j$ is constant in R_S, according to [1.61] we get $\dfrac{d_1\vec{b}_j}{dt} = \vec{\Omega}_{R_1R_S} \times \vec{b}_j$, where

– owing to [2.35]: $\vec{\Omega}_{R_1R_S} = \displaystyle\sum_{i=1}^{n} \vec{\omega}_{1S}^{i}\,\dot{q}_i + \vec{\omega}_{1S}^{t}$,

– from definition [1.38]:

$$\frac{d_1\vec{b}_j}{dt} \equiv \bar{\bar{Q}}_{01}.\frac{d}{dt}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right) = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right)\dot{q}_i + \bar{\bar{Q}}_{01}.\frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right)$$

From this, we can deduce

$$\sum_{i=1}^{n}\left[\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right) - \vec{\omega}_{1S}^{i} \times \vec{b}_j\right]\dot{q}_i + \left[\bar{\bar{Q}}_{01}.\frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right) - \vec{\omega}_{1S}^{t} \times \vec{b}_j\right] = \vec{0}$$

that is, an equation of the form (omitting the fixed index j)

$$\sum_{i=1}^{n}\vec{A}_i(q,t)\dot{q}_i + \vec{A}_t(q,t) = \vec{0} \qquad [2.39]$$

where $\vec{A}_i$ and $\vec{A}_t$ are functions of (q, t), just as $\bar{\bar{Q}}_{01}, \bar{\bar{Q}}_{10}, \vec{b}_j, \vec{\omega}_{1S}^{i}$ and $\vec{\omega}_{1S}^{t}$ are.

To obtain [2.37], let us show that the vectors $\vec{A}_i$ and $\vec{A}_t$ are zero. The reasoning presented below is a little arduous as we must recall that $q = q(t), \dot{q}_i = \dot{q}_i(t)$ and relationship [2.39] hold for any instant t and for any mapping $t \mapsto q(t)$.

Let us arbitrarily fix an instant t and an n-tuple $q = (q_1, \ldots, q_n)$. As [2.39] is valid for any mapping $t \mapsto q(t)$, let us choose this such that, at a given instant t, its value $q(t)$ is equal to the value q that was just chosen and that the derivative $\dot{q}(t)$ is zero. Relationship [2.39] then gives $\vec{A}_t(q,t) = \vec{0}$. As this relationship holds good for any arbitrarily chosen (q,t), we obtain $[2.37]_2$.

Relationship [2.39] now reduces to

$$\sum_{i=1}^{n} \vec{A}_i(q,t)\dot{q}_i = \vec{0} \qquad [2.40]$$

Let us arbitrarily fix an instant t, an n-tuple q and an index $i \in [1,n]$. Once again, as [2.40] is valid for any mapping $t \mapsto q(t)$, let us choose the mapping for which, at the given instant, t, its value $q(t)$ is equal to the value q that was just chosen and for which the derivatives $\dot{q}_i(t)$ are all zero except $\dot{q}_j(t)$. Relationship [2.40] then gives $\vec{A}_i(q,t) = \vec{0}$. As this relationship is valid for (q,t) as well as for an arbitrarily chosen index i, we obtain $[2.37]_1$. ■

Theorem. $\forall$ reference frame R_1, $\forall$ *rigid body* S defining a reference frame R_S, $\forall$ particles p, p' belonging to the rigid body S, whose respective positions are P, P' in R_0, we have the following equalities that hold $\forall t$, $\forall q$:

$$\forall i \in [1,n], \quad \boxed{\bar{\bar{Q}}_{01}.\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{PP'}\right) = \vec{\omega}^i_{1S} \times \overrightarrow{PP'}}$$
$$\boxed{\frac{\partial_{R_1}\overrightarrow{PP'}}{\partial t} \underset{[1.41]}{=} \bar{\bar{Q}}_{01}.\frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\overrightarrow{PP'}\right) = \vec{\omega}^t_{1S} \times \overrightarrow{PP'}} \qquad [2.41]$$

where $\vec{\omega}^i_{1S}$, $\vec{\omega}^t_{1S}$ are the partial angular velocities of S defined in [2.35].

- If we add hypothesis [2.26], then the expression $[2.41]_1$ can be simplified to

$$\forall i \in [1,n], \quad \boxed{\frac{\partial \overrightarrow{PP'}}{\partial q_i} = \vec{\omega}^i_{1S} \times \overrightarrow{PP'}} \qquad [2.42]$$

PROOF. One just has to write $\overrightarrow{PP'} = \sum_{j=1}^{3} c_j \vec{b}_j$, where the components c_j are constants, and to then apply [2.37]–[2.38]. ■

2.9. Velocities in a mechanical system

Let us return to section 1.10 concerning the velocities in a mechanical system to add just one comment on the Eulerian notation [1.65].

Let p be a particle of a mechanical system, $\mathcal{S}$, whose position is $P = pos_{R_0}(p,t)$ in R_0 over the course of time. Using hypothesis [2.26], we have

$$\forall t, \qquad \vec{V}_{R_1\mathcal{S}}(P,t) = \vec{V}_{R_1}(p,t) \underset{[1.47]}{=} \frac{d_{R_1}\overrightarrow{O_1P}}{dt} \underset{[2.27]}{=} \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1P}}{\partial q_i}\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}$$

On the other hand, for any point $A \in \mathcal{E}$:

$$\vec{V}_{R_1\mathcal{S}}(A,t) \neq \frac{d_{R_1}\overrightarrow{O_1A}}{dt} = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1A}}{\partial q_i}\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1A}}{\partial t}$$

The equality occurs only when A is a point *attached to the system* $\mathcal{S}$, that is, when A denotes the position of the *same* particle of the system over the course of time.

2.10. Parameterized velocity of a particle

The objectives set here and for the longer term are as follows:

– We will define a new function called the *parameterized velocity* with respect to R_1.

– With the help of this parameterized velocity, we will establish the so-called kinematic Lagrange's formulae [2.52] and we will define the parameterized kinetic energy [2.54] of the mechanical system with respect to R_1, which is an essential ingredient in analytical mechanics.

– This will enable us to calculate, in Chapter 5, the VP of the quantities of acceleration with respect to R_1, or more precisely, the coefficients denoted by C_i in [5.39], which constitutes the left-hand side of Lagrange's equations [6.2].

The reference frame R_1 in which we calculate the VP of the quantities of acceleration is the reference frame in which we write the principle of VP [5.1]. Now, in practice it turns out that this reference frame R_1 (whether Galilean or not) is such that the rotation tensor $\bar{\bar{Q}}_{01}$ does *not* depend on q. Thus, unlike the discussion in section 2.6 here it is useless to consider the case when $\bar{\bar{Q}}_{01}$ depends on q when studying the above-mentioned ingredients: namely, the parameterized velocity, the kinematic Lagrange's formulae, the parameterized kinetic energy and the quantities of acceleration. We will, thus, study them by using hypothesis [2.26] from the beginning. This hypothesis is reviewed below:

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

2.10.1. *Definition*

Using hypothesis [2.26], we have expression [2.27] for the velocity $\vec{V}_{R_1}(p,t)$ and we will use it to define a new function – improperly denoted by $\vec{V}_{R_1}$ – which depends on the new variables $(q,\dot{q},t)$:

Definition. Let p be a particle whose position in R_1 is given by [2.21]: $P = P(q,t)$.

The parameterized velocity of p with respect to R_1, denoted $\vec{V}_{R_1}(q,\dot{q},t)$, is defined as

$$\forall(q,\dot{q},t), \quad \boxed{\vec{V}_{R_1}(q,\dot{q},t) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q,t)\,\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}(q,t)} \quad \text{where} \quad \boxed{\begin{array}{l} q \equiv (q_1,\ldots,q_n) \\ \dot{q} \equiv (\dot{q}_1,\ldots,\dot{q}_n) \end{array}} \qquad [2.43]$$

and O_1 is a fixed point in R_1. If we adopt hypothesis [2.33], which is a little stronger than hypothesis [2.26], then the above expression can be simplified a little to:

$$\boxed{\vec{V}_{R_1}(q,\dot{q},t) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{P}}{\partial q_i}(q,t)\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}(q,t)} \qquad [2.43b]$$

The new function $\vec{V}_{R_1}(q,\dot{q},t)$ is special and must be understood as follows:

– It is defined as a function of $2n+1$ *independent* variables $q,\dot{q},t$. It is, therefore, important to treat $q,\dot{q},t$ in [2.43] *as variables that are mutually independent*.

Clearly, it would be more logical to replace $\dot{q}_i$ with other letters, r_i for example. However, we have chosen not to do this systematically in order to avoid multiplying the number of notations used. We will retain the notation $\dot{q}_i$, bearing in mind that here $\dot{q}_i$ should not be treated as the time derivative of q_i, but instead as a variable *independent* of q_i.

– The variables $q, \dot{q}, t$ in [2.43] may take *arbitrary* values regardless of any motion or velocity: q is *not* obtained through $t \mapsto q(t)$ and $\dot{q}$ is *not* obtained through $t \mapsto \frac{dq}{dt}(t)$.

Thus, the parameterized velocity is *not necessarily* equal to the velocity of a particle with respect to a reference frame. It is only when q is replaced by a function $q(t)$ and $\dot{q}$ by $\frac{dq}{dt}(t)$ that $\vec{V}_{R_1}(q, \dot{q}, t)$ is equal to the velocity $\vec{V}_{R_1}(p, t)$ in [2.27].

In order to avoid multiplying the number of notations used, we have used the same symbol, $\vec{V}_{R_1}$, to designate both the classical velocity $\vec{V}_{R_1}(p, t)$ and the parameterized velocity $\vec{V}_{R_1}(q, \dot{q}, t)$. However, there is no possible confusion since the difference in arguments enables one to easily distinguish between the two velocities.

Strictly speaking, we should write $\vec{V}_{R_1}(p; q, \dot{q}, t)$, with the particle p among the arguments. However, the context is often clear enough for us to be able to drop variable p so as to simplify the writing.

2.10.2. *Practical calculation of the parameterized velocity*

Being able to calculate the parameterized velocity will enable one to calculate the parameterized kinetic energy $E^c_{R_1 S}(q, \dot{q}, t)$ defined further in [2.54].

By bringing together the two relationships [2.27] and [2.43], we can see that the real velocity $\vec{V}_{R_1}(p, t)$ and the parameterized velocity $\vec{V}_{R_1}(q, \dot{q}, t)$ are very close. In order to calculate a velocity by means of one of these relationships, we start by calculating

$$\sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1 P}}{\partial q_i}(q, t)\, \dot{q}_i + \frac{\partial_{R_1} \overrightarrow{O_1 P}}{\partial t}(q, t) \qquad [2.44]$$

If we stop with this, we obtain the parameterized velocity $\vec{V}_{R_1}(q, \dot{q}, t)$. If we wish to obtain the real velocity $\vec{V}_{R_1}(p, t)$, we must carry out an additional operation, namely replacing q by $q(t)$ and $\dot{q}$ by $\frac{dq}{dt}(t)$. We thus calculate the parameterized velocity every time without explicitly (stating it).

Furthermore, in practice, when we calculate the real velocity, we systematically drop the additional operation mentioned above. *It becomes implicit*, because not only does it complicate the equations, but it is also so "evident" to the physicist's mind that stating the operation adds no real value.

In practice, the real and parameterized velocities have the *same* expression.

EXAMPLE. Let us consider a disc S of C and radius R, which moves in the plane $O\vec{x}_0\vec{y}_0$ while staying in *frictionless* contact with the axis $O\vec{x}_0$, at the point I (Figure 2.7).

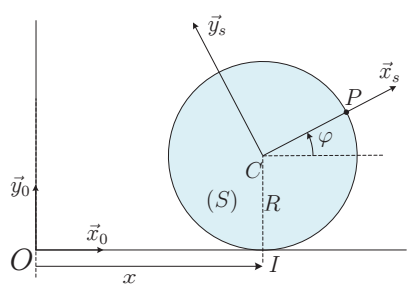


Figure 2.7. *Disc rolling on an axis*

The following parameterization is chosen:

– primitive parameters: the coordinates x, y of center C, the angle of rotation φ of the disc, defined by $\varphi = (\widehat{\vec{x}_0, \vec{x}_S})$ measured around $\vec{z}_0$, $\vec{x}_S$ being a vector that is attached to S;

– primitive constraint equation: $y = R$;

– retained parameters: $q = (x, \varphi)$;

– complementary constraint equation: none (frictionless contact at point I).

Let us consider the particle p of S, located on the disc boundary and on the axis $C\vec{x}_S$ attached to S:

$$\overrightarrow{OP} = x\vec{x}_0 + R\vec{y}_0 + R\vec{x}_S(\varphi) \qquad \text{with } \vec{x}_S(\varphi) = \cos\varphi\vec{x}_0 + \sin\varphi\vec{y}_0$$

Applying [2.44], with $R_1 = R_0$ and $O_1 = O$, gives the parameterized velocity $\vec{V}_{R_1}(q, \dot{q}, t)$:

$$\vec{V}_{R_1}(q, \dot{q}, t) = \dot{x}\vec{x}_0 + R\dot{\varphi}\vec{y}_S(\varphi) \qquad \text{with } \vec{y}_S(\varphi) = \vec{z}_0 \times \vec{x}_S(\varphi)$$

The real velocity is obtained by replacing $\dot{x} = \dot{x}(t) \equiv \frac{dx(t)}{dt}$ and $\varphi = \varphi(t)$ in the previous expression:

$$\vec{V}_{R_1}(p, t) = \dot{x}(t)\vec{x}_0 + R\dot{\varphi}(t)\vec{y}_S(\varphi(t))$$

In practice, we never write this last expression, which is cumbersome, and it suffices to write $\vec{V}_{R_1}(p, t) = \dot{x}\vec{x}_0 + R\dot{\varphi}\vec{y}_S(\varphi)$, which is the same expression as for the parameterized velocity. ■

2.11. Parameterized velocities in a rigid body

The following relationship can be shown between the parameterized velocities in a rigid body, using $[2.41]_2$ and [2.42]:

Theorem and definition. $\forall t$, $\forall$ reference frame R_1, $\forall$ rigid body S defining a reference frame R_S, $\forall(q, \dot{q}, t)$, $\forall$ particles p, p' belonging to the rigid body S, with the respective positions $P = P(q, t)$, $P' = P'(q, t)$ in R_0, we have

$$\boxed{\vec{V}_{R_1}(p'; q, \dot{q}, t) = \vec{V}_{R_1}(p; q, \dot{q}, t) + \vec{\Omega}_{R_1 R_S}(q, \dot{q}, t) \times \overrightarrow{PP'}} \qquad [2.45]$$

where $\vec{\Omega}_{R_1 S}(q, \dot{q}, t)$, referred to as the *parameterized angular velocity of S in R_1*, is defined similarly to [2.35]:

$$\boxed{\vec{\Omega}_{R_1 R_S}(q, \dot{q}, t) = \sum_{i=1}^{n} \vec{\omega}^i_{1S}(q, t)\, \dot{q}_i + \vec{\omega}^t_{1S}(q, t)} \qquad [2.46]$$

Parameterized velocity field

The Eulerian notation for parameterized velocities is defined in a manner similar to [1.62]:

Eulerian notation. Let S be a rigid body and A a point such that $A \in pos_{R_0}(S, t)$. We denote

$$\boxed{\vec{V}_{R_1 S}(A; q, \dot{q}, t) \equiv \text{ the parameterized velocity with respect to } R_1 \text{ of the particle of } S \text{ passing through point } A \text{ at instant } t} \qquad [2.47]$$

When using the Eulerian notation, the particle is not known by its name but by its position at the instant considered. In general, the particle is *not* the same over time.

Using the Eulerian notation [2.47], we can define *the field of parameterized velocities* $V_{R_1S}(.;q,\dot{q},t)$ of a rigid body S with respect to R_1 in a manner that is similar to what is done for real velocities.

Parameterized velocity fields on a rigid body

In the case of a *rigid body*, the following result, similar to [1.63], can be proved:

Theorem. $\forall t, \forall R_1, \forall$ *rigid body* S defining a reference frame R_S, $\forall(q,\dot{q},t)$, $\forall A, B \in pos_{R_0}(S,t) \subset \mathcal{E}$,

$$\boxed{\vec{V}_{R_1S}(B;q,\dot{q},t) = \vec{V}_{R_1S}(A;q,\dot{q},t) + \vec{\Omega}_{R_1R_S}(q,\dot{q},t) \times \overrightarrow{AB}} \qquad [2.48]$$

Thus, $\forall t$, the field of parameterized velocities $V_{R_1S}(.;q,\dot{q},t)$, satisfies the well-known velocity relationship in a rigid body, characterized by vector $\vec{\Omega}_{R_1R_S}(q,\dot{q},t)$ defined in [2.46].

Once again, it can be seen that when we calculate the real velocity using formula [1.63], we always unknowingly go through the parameterized velocity [2.48]. As concerns the additional operation, which consists of replacing q with $q(t)$ and $\dot{q}$ with $\frac{dq}{dt}(t)$, once again we drop it from the final written form here.

2.12. Parameterized velocities in a mechanical system

Field of parameterized velocities

The Eulerian notation of parameterized velocities in a mechanical system is defined in a manner similar to [2.47] in a rigid body:

Eulerian notation. Let $\mathcal{S}$ be a mechanical system and A a point such that $A \in pos_{R_0}(\mathcal{S},t)$. We denote

$$\boxed{\vec{V}_{R_1\mathcal{S}}(A;q,\dot{q},t) \equiv \text{the parameterized velocity with respect to } R_1 \text{ of the particle of } \mathcal{S} \text{ passing through point } A \text{ at instant } t} \qquad [2.49]$$

When using this Eulerian notation, the particle is not known by its name but by its position at the instant considered. In general, the particle is *not* the same over time.

Using the Eulerian notation [2.49], we can define the *field of parameterized velocities* $V_{R_1\mathcal{S}}(.;q,\dot{q},t)$ of a mechanical system $\mathcal{S}$ with respect to R_1 in a manner similar to that defined for a rigid body.

Summary

In summary, the real and parameterized velocities are calculated using similar formulae and, in practice, we obtain the *same expression* for both of them.

This has significant consequences on the parameterized kinetic energy $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$, which will be defined in [2.54] and which is an essential ingredient of Lagrange's equations:

(i) the parameterized kinetic energy is also calculated using formulae similar to those used for classical kinetic energy $E^c_{R_1\mathcal{S}}(t)$, which are well known in Newtonian mechanics;

(ii) in practice, we obtain *the same expression* for both kinetic energies.

2.13. Lagrange's kinematic formula

The results established in this section are derived from the parameterized velocity [2.43], which is defined using hypothesis [2.26], which we reproduce below:

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

Lemma. The parameterized velocity $\vec{V}_{R_1}(q, \dot{q}, t)$ defined in [2.43] verifies the following relations, valid for *all values taken by* $(q, \dot{q}, t)$:

$$\forall i \in [1, n], \quad \boxed{\frac{\partial \vec{V}_{R_1}}{\partial \dot{q}_i}(q, \dot{q}, t) = \frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q, t)}$$
$$\text{and} \quad \boxed{\frac{\partial \vec{V}_{R_1}}{\partial q_i}(q, \dot{q}, t) = \sum_{j=1}^{n} \frac{\partial}{\partial q_j}\left(\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q, t)\right)\dot{q}_j + \frac{\partial_{R_1}}{\partial t}\left(\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q, t)\right)} \qquad [2.50]$$

where, as in [2.27], O_1 is a fixed point in R_1.

PROOF. The first equality follows immediately from [2.43]. In order to prove the second equality, let us rewrite [2.43] by replacing the dumb index i by j and then derive the result with respect to q_i:

$$\frac{\partial \vec{V}_{R_1}}{\partial q_i}(q, \dot{q}, t) = \sum_{j=1}^{n} \frac{\partial}{\partial q_i}\left(\frac{\partial \overrightarrow{O_1P}}{\partial q_j}(q, t)\right)\dot{q}_j + \frac{\partial}{\partial q_i}\left(\frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}(q, t)\right)$$

According to hypothesis [2.22], the mapping $(q, t) \mapsto \overrightarrow{O_1P}(q, t)$ belongs to class C^2. It is, thus, possible to interchange the order of the derivatives on the right-hand side to obtain the second equality in [2.50]. ■

Theorem.

$$\forall i \in [1, n], \forall t, \quad \boxed{\frac{\partial \vec{V}_{R_1}}{\partial \dot{q}_i}(q(t), \dot{q}(t), t) = \frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t), t)}$$
$$\text{and} \quad \boxed{\frac{\partial \vec{V}_{R_1}}{\partial q_i}(q(t), \dot{q}(t), t) = \frac{d_{R_1}}{dt}\left(\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t), t)\right)} \qquad [2.51]$$

PROOF. Let us apply [2.50] with q replaced by a function $q(t)$ and $\dot{q}$ by $\frac{dq}{dt}(t)$. Relationship $[2.50]_1$ immediately gives $[2.51]_1$, while $[2.50]_2$ gives

$$\frac{\partial \vec{V}_{R_1}}{\partial q_i}(q(t), \dot{q}(t), t) = \sum_{j=1}^{n} \frac{\partial}{\partial q_j}\left(\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t), t)\right)\dot{q}_j(t) + \frac{\partial_{R_1}}{\partial t}\left(\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t), t)\right)$$

that is, $[2.51]_2$, taking into account hypothesis [2.26]. ■

The following result can be derived from [2.51] and it will be used to calculate the VP of the quantities of acceleration in section 5.9:

Theorem. Lagrange's kinematic formula.

The projections of the acceleration vector onto the vectors $\frac{\partial \overrightarrow{O_1P}}{\partial q_i}$ are given by

$$\forall i \in [1,n], \quad \boxed{\begin{aligned} &\vec{\Gamma}_{R_1}(p,t) \,.\, \frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t),t) = \\ &\frac{d}{dt}\frac{\partial\left(\frac{1}{2}\vec{V}^2_{R_1}(q,\dot{q},t)\right)}{\partial \dot{q}_i}(q(t),\dot{q}(t),t) - \frac{\partial\left(\frac{1}{2}\vec{V}^2_{R_1}(q,\dot{q},t)\right)}{\partial q_i}(q(t),\dot{q}(t),t)\end{aligned}} \qquad [2.52a]$$

or, in shortened form:

$$\forall i, \qquad \boxed{\vec{\Gamma}_{R_1} \,.\, \frac{\partial \overrightarrow{O_1P}}{\partial q_i} = \frac{d}{dt}\frac{\partial\left(\frac{1}{2}\vec{V}^2_{R_1}\right)}{\partial \dot{q}_i} - \frac{\partial\left(\frac{1}{2}\vec{V}^2_{R_1}\right)}{\partial q_i}} \qquad [2.52b]$$

PROOF. We use formula [1.45]:

$$\forall \vec{a}, \vec{b}, \qquad \frac{d}{dt}(\vec{a}.\vec{b}) = \frac{d_{R_1}\vec{a}}{dt} \,.\, \vec{b} + \vec{a} \,.\, \frac{d_{R_1}\vec{b}}{dt}$$

which gives

$$\begin{aligned} \frac{d}{dt}\left[\vec{V}_{R_1}(p,t).\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t),t)\right] &\underset{[2.51]a}{=} \frac{d}{dt}\left[\vec{V}_{R_1}(p,t).\frac{\partial \vec{V}_{R_1}}{\partial \dot{q}_i}(q(t),\dot{q}(t),t)\right] \\ &= \underbrace{\frac{d_{R_1}}{dt}\vec{V}_{R_1}(p,t)}_{\underset{[1.66]}{=}\ \vec{\Gamma}_{R_1}(p,t)} \,.\, \frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t),t) + \vec{V}_{R_1}(p,t). \underbrace{\frac{d_{R_1}}{dt}\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t),t)}_{\underset{[2.51]b}{=}\ \frac{\partial \vec{V}_{R_1}}{\partial q_i}(q(t),\dot{q}(t),t)} \end{aligned}$$

Hence

$$\vec{\Gamma}_{R_1}(p,t).\frac{\partial \overrightarrow{O_1P}}{\partial q_i}(q(t),t) = \frac{d}{dt}\left[\vec{V}_{R_1}(p,t).\frac{\partial \vec{V}_{R_1}}{\partial \dot{q}_i}(q(t),\dot{q}(t),t)\right] - \vec{V}_{R_1}(p,t).\frac{\partial \vec{V}_{R_1}}{\partial q_i}(q(t),\dot{q}(t),t)$$

What then remains is rewriting $\vec{V}_{R_1}(p,t) = \vec{V}_{R_1}(q(t),\dot{q}(t),t)$. ■

Note. The following important points must be kept in mind:

1. In accordance with definition [2.43], $\frac{1}{2}\vec{V}^2_{R_1}(q,\dot{q},t)$, which appears in [2.52], must be treated not as a composite function of t, but as a function of $2n+1$ *independent* variables. The expression for $\vec{V}^2_{R_1}$ is given by [2.43]:

$$\vec{V}^2_{R_1}(q,\dot{q},t) = \sum_i\sum_j \frac{\partial \overrightarrow{O_1P}}{\partial q_i}\frac{\partial \overrightarrow{O_1P}}{\partial q_j}\dot{q}_i\dot{q}_j + 2\sum_i \frac{\partial \overrightarrow{O_1P}}{\partial q_i}\frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}\dot{q}_i + \left(\frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}\right)^2 \qquad [2.53]$$

2. It is, therefore, important to apply the Lagrange kinematic formula [2.52] by carrying out the following operations *in the given order*:

(a) Calculate the derivatives $\dfrac{\partial\left(\frac{1}{2}\vec{V}_{R_1}^2\right)}{\partial\dot{q}_i}$, $\dfrac{\partial\left(\frac{1}{2}\vec{V}_{R_1}^2\right)}{\partial q_i}$ with respect to variables $\dot{q}_i$, q_i by considering these variables to be *independent*.

(b) Next, in the expressions obtained, replace q with $q(t)$ and $\dot{q}$ with the derivative $\dot{q}(t) = \frac{dq(t)}{dt}$.

(c) Finally, calculate the derivative $\frac{d}{dt}$ as the derivative of the composite function of t thus formed.

The Lagrange kinematic formula shows that knowing just the function $\frac{1}{2}\vec{V}_{R_1}^2(q,\dot{q},t)$ of $2n+1$ variables $q,\dot{q},t$ is enough to know the projections of the acceleration $\vec{\Gamma}_{R_1}(p,t)$ on the vectors $\frac{\partial\overrightarrow{O_1P}}{\partial q_i}$.

– In space (three independent parameters), formula [2.52] completely determines $\vec{\Gamma}_{R_1}(p,t)$ through its three (covariant) components in the local basis at $P(t)$.

– If the trajectory lies, *a priori*, on a surface (two independent parameters), formula [2.52] only gives the projection of $\vec{\Gamma}_{R_1}(p,t)$ on the tangent plane at $P(t)$.

– If the trajectory is, *a priori*, part of a curve (one independent parameter), formula [2.52] only gives the projection of $\vec{\Gamma}_{R_1}(p,t)$ on the tangent at $P(t)$.

2.14. Parameterized kinetic energy

For all practical purposes, let us introduce a new function that is called the parameterized kinetic energy and that *resembles* the kinetic energy of the system $\mathcal{S}$ with respect to R_1. This function is based on the parameterized velocity $\vec{V}_{R_1}(q,\dot{q},t)$, which is defined in [2.43] using hypothesis [2.26]. The hypothesis is reproduced below for reference:

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

Definition. *The parameterized kinetic energy of the system $\mathcal{S}$ with respect to R_1*, denoted by $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$, is defined as

$$\forall(q,\dot{q},t),\qquad \boxed{E^c_{R_1\mathcal{S}}(q,\dot{q},t) \equiv \frac{1}{2}\int_{\mathcal{S}} \vec{V}_{R_1}^2(q,\dot{q},t)\mathrm{d}m} \qquad [2.54]$$

The new function $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$ is special and we must understand it in the same way as the parameterized velocity $\vec{V}_{R_1}(q,\dot{q},t)$ defined in [2.43]:

– $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$ is defined as a function of $2n+1$ *independent* variables $q,\dot{q},t$. It is, therefore, important to treat $q,\dot{q},t$ in [2.54] *as variables that are mutually independent*.

  To be rigorous, one should replace $\dot{q}_i$ with other letters, r_i, for example. However, this has not been done simply to avoid multiplying the number of notations used. We will retain the notation $\dot{q}_i$, keeping in mind that $\dot{q}_i$ here does not signify the time derivative of q_i, but denotes a variable *independent* of q_i.

– The variables $q,\dot{q},t$ in [2.54] may take *arbitrary* values, regardless of any motion or velocity: q is *not* obtained through $t \mapsto q(t)$ nor $\dot{q}$ through $t \mapsto \frac{dq}{dt}(t)$.

  Thus, the parameterized kinetic energy is *not necessarily* equal to the kinetic energy of the system $\mathcal{S}$ with respect to R_1 at t. It is only when we replace q by the function $q(t)$ and

$\dot{q}$ by $\frac{dq}{dt}(t)$ that $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$ is equal to the kinetic energy, $E^c_{R_1\mathcal{S}}(t)$, of the system $\mathcal{S}$ with respect to R_1 at t.

By extending the results in section 2.10.2, it appears that in order to calculate the parameterized kinetic energy $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$, one has just to go back to the well-known formulae in Newtonian mechanics for the classical kinetic energy $E^c_{R_1\mathcal{S}}(t)$, and adapt these for the parameterized kinetic energy.

As a matter of fact, when calculating the classical kinetic energy, we always begin by, quite unknowingly, obtaining the parameterized kinetic energy. Only after obtaining the expression for $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$ do we deduce the expression for $E^c_{R_1\mathcal{S}}(t)$, replacing q with $q(t)$ and $\dot{q}$ with $\frac{dq}{dt}(t)$. In practice, this supplementary operation is dropped so as to simplify the mathematical expressions, with the result that the two kinetic energies have *the same expression*.

We can get an explicit expression for the parameterized kinetic energy $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$ by means of [2.53]:

$$\boxed{2E^c_{R_1\mathcal{S}}(q,\dot{q},t) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(q,t)\,\dot{q}_i\dot{q}_j + 2\sum_{i=1}^{n} b_i(q,t)\,\dot{q}_i + c(q,t)} \qquad [2.55]$$

with

$$\boxed{a_{ij} = \int_{\mathcal{S}} \frac{\partial\overrightarrow{O_1P}}{\partial q_i}\cdot\frac{\partial\overrightarrow{O_1P}}{\partial q_j}dm \qquad b_i = \int_{\mathcal{S}} \frac{\partial\overrightarrow{O_1P}}{\partial q_i}\cdot\frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}dm \qquad c = \int_{\mathcal{S}}\left(\frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}\right)^2 dm} \qquad [2.56]$$

Expression [2.55] leads to the following decomposition:

Definition. The parameterized kinetic energy $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$ is decomposed as follows:

$$\boxed{E^c_{R_1\mathcal{S}} = E^{c(2)}_{R_1\mathcal{S}} + E^{c(1)}_{R_1\mathcal{S}} + E^{c(0)}_{R_1\mathcal{S}}} \qquad [2.57]$$

where the functions $E^{c(2)}_{R_1\mathcal{S}}$, $E^{c(1)}_{R_1\mathcal{S}}$ and $E^{c(0)}_{R_1\mathcal{S}}$, respectively, are defined as the parts of $E^c_{R_1\mathcal{S}}$ which are of second, first and zero degree, respectively, with respect to the derivatives $\dot{q}_i$:

$$\boxed{E^{c(2)}_{R_1\mathcal{S}} \equiv \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(q,t)\dot{q}_i\dot{q}_j} \qquad \boxed{E^{c(1)}_{R_1\mathcal{S}} \equiv \sum_{i=1}^{n} b_i(q,t)\dot{q}_i} \qquad \boxed{E^{c(0)}_{R_1\mathcal{S}}(q,t) \equiv \frac{1}{2}c(q,t)} \geq 0 \qquad [2.58]$$

In other words, $E^{c(2)}_{R_1\mathcal{S}}$ is the part of $E^c_{R_1\mathcal{S}}$, which is quadratic in the $\dot{q}_i$, $E^{c(1)}_{R_1\mathcal{S}}$ is the part that is linear and $E^{c(0)}_{R_1\mathcal{S}}$ is the part that is independent of the $\dot{q}_i$.

In the case when $\dfrac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t} = \vec{0}$, we have more precise information on the coefficients a_{ij}, b_j, c and the energies $E^{c(2)}_{R_1\mathcal{S}}, E^{c(1)}_{R_1\mathcal{S}}, E^{c(0)}_{R_1\mathcal{S}}$:

Theorem.

HYPOTHESES:

(i) Let us recall that we have adopted hypothesis [2.26]: the rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

(ii) Let us assume, additionally, that $\dfrac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t} = \vec{0}$.

Then

$$\forall i \in [1,n], \boxed{b_i = 0}, \boxed{c = 0} \Rightarrow \boxed{E^{c(0)}_{R_1\mathcal{S}}(q,t) = E^{c(1)}_{R_1\mathcal{S}}(q,t) = 0} \Rightarrow \boxed{E^{c}_{R_1\mathcal{S}} = E^{c(2)}_{R_1\mathcal{S}}}$$
$$\forall i,j \in [1,n], \boxed{a_{ij} = a_{ij}(q) \text{ independent of } t} \Rightarrow \boxed{E^{c(2)}_{R_1\mathcal{S}} = E^{c(2)}_{R_1\mathcal{S}}(q,\dot{q}) \text{ independent of } t} \qquad [2.59]$$

PROOF. As hypothesis (ii) immediately implies $[2.59]_1$, let us show that a_{ij} are independent of t. As $\overrightarrow{O_1P} = \bar{\bar{Q}}_{01}.\overrightarrow{OP^{(1)}}$ (see [2.24]), we have

$$\frac{\partial\overrightarrow{O_1P}}{\partial q_i}.\frac{\partial\overrightarrow{O_1P}}{\partial q_j} = \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{01}.\overrightarrow{OP^{(1)}}\right).\frac{\partial}{\partial q_j}\left(\bar{\bar{Q}}_{01}.\overrightarrow{OP^{(1)}}\right)$$

According to hypothesis (i), we can move $\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{01}(t)$, independent of q, outside the derivatives:

$$\frac{\partial\overrightarrow{O_1P}}{\partial q_i}.\frac{\partial\overrightarrow{O_1P}}{\partial q_j} = \frac{\partial\overrightarrow{OP^{(1)}}}{\partial q_i}\underbrace{\left(\bar{\bar{Q}}^T_{01}.\bar{\bar{Q}}_{01}\right)}_{\bar{\bar{I}}}.\frac{\partial\overrightarrow{OP^{(1)}}}{\partial q_j} = \frac{\partial\overrightarrow{OP^{(1)}}}{\partial q_i}.\frac{\partial\overrightarrow{OP^{(1)}}}{\partial q_j}$$

Furthermore, taking into account definition [1.41], hypothesis (ii) can be written as

$$\vec{0} = \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t} \equiv \bar{\bar{Q}}_{01}.\frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right), \text{ that is } \vec{0} = \frac{\partial}{\partial t}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right) = \frac{\partial\overrightarrow{OP^{(1)}}}{\partial t},$$

which signifies that $\dfrac{\partial\overrightarrow{OP^{(1)}}}{\partial t}$ does not explicitly depend on t. ■

In the particular case when we choose $R_0 = R_1$, hypothesis (ii) becomes $\dfrac{\partial\overrightarrow{O_1P}}{\partial t} = \vec{0}$ and the result for the coefficients a_{ij} is straightforward.

3

Efforts

3.1. Forces

Forces can be divided into two categories based on the distance between the body exerting the force and the particle upon which the force acts:

1. *Contact forces* are mutually exerted between the particles in contact from two different bodies. These forces act on the surface of the contacting bodies and prevent the bodies from penetrating each other.

2. *At-a-distance forces*. Unlike contact forces, an at-a-distance force arises from an agent located at a non-zero distance from the particle upon which the force acts. Gravitational force (thus weight) and magnetic force are examples of at-a-distance forces.

In order to write equations in mechanics, however, we prefer classifying forces according to the way they are distributed:

1. *A distributed force* is a force field applied over a set of particles (not necessarily the same set over time) whose position is a volume, a surface or a curve in space.

   Depending on the case, we shall define a volume force, a surface force or a line force. In all cases, we also define a *mass force*, for which the unit is $\mathrm{N/kg}$.

2. In the case when a distributed force is applied over a domain that is considered to be sufficiently small with regard to the problem studied, the distributed force is modeled as a concentrated force. A *concentrated force* is a finite force applied on a particle (not necessarily the same particle over time).

In the case of a *discrete* system (i.e., a system formed of a finite number of particles), any force applied to a particle of the system is *necessarily a concentrated force*.

In order to introduce the notations used for forces, let us consider the forces exerted by a system $\mathcal{S}'$ on a system $\mathcal{S}$. These forces may be at-a-distance forces or constraint forces, either through direct contact or through an intermediate device such as a spring and a connecting rod.

1. Let there be a concentrated force exerted by $\mathcal{S}'$ on a particle a of $\mathcal{S}$, whose position in R_0 is $A = pos_{R_0}(a,t)$. This force is denoted by $\boxed{\vec{F}_{\mathcal{S}'\to\mathcal{S}}(A,t)}$, which is to be read as, "the concentrated force (exerted by $\mathcal{S}'$ on $\mathcal{S}$) acting at point A and at instant t". The point $A \in \mathcal{E}$ is called the point of application of the force.

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

If the variable of time is implied, we write $\boxed{\vec{F}_{\mathcal{S}'\to\mathcal{S}}(A)}$. If there is no need to specify the agent $\mathcal{S}'$ of the force, we write $\boxed{\vec{F}_{\to\mathcal{S}}(A)}$. Further, if the context is clear enough, such that we do not need to specify the system $\mathcal{S}$, which is subjected to the force, we write $\boxed{\vec{F}(A)}$. The point of application A may even be absent if it is clearly identified in the considered problem.

2. Let there be a mass force exerted by $\mathcal{S}'$ on $\mathcal{S}$ (or a part $\mathcal{S}_e$ of $\mathcal{S}$). We then write $\boxed{\vec{f}_{\mathcal{S}'\to\mathcal{S}}(A,t)}$ *the value of the force field* (or the *force distribution*) $\vec{f}_{\mathcal{S}'\to\mathcal{S}}(t)$ at a current point $A \in pos_{R_0}(\mathcal{S}_e, t)$.

   As in the case of a concentrated force, we may use abbreviated notations whenever the context allows this: $\boxed{\vec{f}_{\mathcal{S}'\to\mathcal{S}}(A)}$, $\boxed{\vec{f}_{\to\mathcal{S}}(A)}$ or $\boxed{\vec{f}(A)}$.

   Knowing the mass force $\vec{f}$, it is possible to derive the volume force (respectively, surface, line force) using $\rho\vec{f}$ (respectively, $\rho_\Sigma\vec{f}$, $\rho_L\vec{f}$), where ρ (respectively, ρ_Σ, ρ_L) denotes the density of the body (respectively, the mass per unit area, length).

EXAMPLES OF NOTATIONS.

1. Let us consider a disc S in contact with a plane S' (Figure 3.1 (left)). At the contact point A, there are two forces – one exerted by S' on S, the other by S on S' – both of which have the same point of application A but are, in fact, exerted on two *different* particles belonging, respectively, to the two bodies, and they are *opposite*.

   In such a situation, the shortened notation $\vec{F}(A,t)$ used to denote the contact force exerted by S' on S is ambiguous. It is thus necessary to use the notation $\vec{F}_{S'\to S}(A,t)$ or $\vec{F}_{\to S}(A,t)$, which will clearly indicate that the force acts on S.

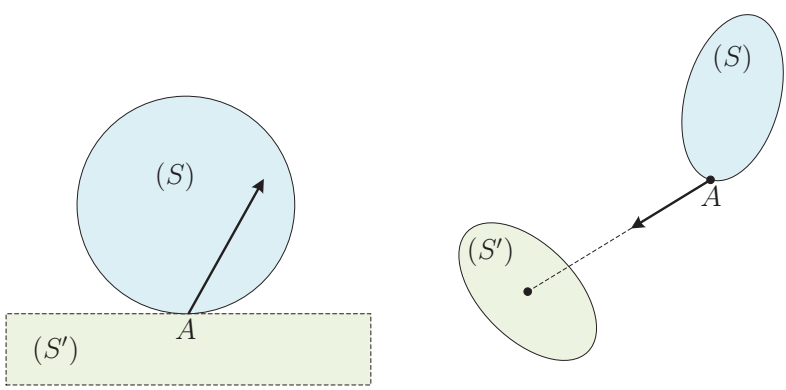


Figure 3.1. *Examples of forces*

2. Let us consider a rigid body S connected to a support S' by a string (Figure 3.1 (right)). The attachment point A of the string on S is the position of a particle of S, and the force exerted on S may be denoted by $\vec{F}_{string\to S}(A,t)$ or $\vec{F}_{S'\to S}(A,t)$ or, in the abbreviated form, $\vec{F}(A,t)$ without any risk of ambiguity. ■

It is important to keep in mind that a force is applied on *the matter* (*one or more particles*) of a system and that a force is not applied to a point (of space $\mathcal{E}$). Thus, when we say, for example, that a concentrated force is applied to a point A, it must be understood that the force is applied to a particle with position A. Similarly, when we say that a volume force is applied on a region, it is understood that the force is applied to the particles whose positions lie in the interior of this region.

When adopting the above notations, for instance $\vec{F}_{\mathcal{S}'\to\mathcal{S}}(A,t)$, we have chosen to focus on the *point of application* of the force, rather than the particle on which the force is exerted. These notations are precise enough to write the equations in mechanics, for example to express the moment field due to a system of forces.

3.2. Torque

Torques can be divided into two categories based on the distance between the body exerting the torque and the particle upon which the torque acts:

1. *Contact torques*, which are mutually exerted between the particles in contact that belong to different bodies. In the case of one rigid body rolling over another, it may prove necessary to introduce contact torques to model the resistance to the rolling and the pivoting.

2. *Action-at-a-distance torques*, which, unlike contact torques, arise from an agent situated at a non-zero distance from the particle upon which the torque acts. A magnetic field entails a distributed at-a-distance torque within magnetic bodies that have their own magnetization.

In practice, as with forces, we prefer classifying torques according to their distribution mode:

1. A *concentrated torque* is a finite torque applied to a particle of a continuous system (the particle is not necessarily the same over time).

2. A *distributed torque* is a torque field applied over a set of particles whose position occupies a volume, a surface or a curve in space. Depending on the case, we define either volume torque, surface torque or line torque and, in all cases, *a mass torque,* which has the unit of $\mathrm{Nm/kg}$.

The torque exerted on an isolated particle must be *zero*. A concentrated torque can only be applied on a particle belonging to a continuous system. It *cannot* be applied on an isolated particle.

The notations used for torques are similar to the notations used for forces. Let us consider the torques exerted by a system $\mathcal{S}'$ on a system $\mathcal{S}$. These torques may be at-a-distance or constraint torques, either through direct contact or through an intermediate organ such as a spiral spring.

1. Let there be a torque exerted by $\mathcal{S}'$ on a particle a of $\mathcal{S}$, whose position in R_0 is $A = pos_{R_0}(a,t)$. It is denoted by $\boxed{\vec{C}_{\mathcal{S}'\to\mathcal{S}}(A,t)}$, which is read as "the concentrated torque (exerted by $\mathcal{S}'$ on $\mathcal{S}$) acting at point A and at instant t". The point $A \in \mathcal{E}$ is called the point of application of the torque. Even if the point of application of a torque on a *rigid body* has no effect on the mechanics equations, specifying this point in the notations enables one to clearly identify where the torque is physically applied.

   If the time variable can be made implicit, we write $\boxed{\vec{C}_{\mathcal{S}'\to\mathcal{S}}(A)}$. If there is no need to specify the agent $\mathcal{S}'$ of the torque, we can write $\boxed{\vec{C}_{\mathcal{S}'\to\mathcal{S}}(A)}$. If, moreover, the context is

clear enough, such that we do not need to specify the system $\mathcal{S}$ that is subjected to the torque, we write $\boxed{\vec{C}(A)}$. The point of application A may even be absent if it is clearly identified in the considered problem.

2. Let there be a mass torque exerted by $\mathcal{S}'$ on $\mathcal{S}$ (or a part $\mathcal{S}_e$ of $\mathcal{S}$). We write $\boxed{\vec{c}_{\mathcal{S}'\to\mathcal{S}}(A,t)}$, which is *the value of the torque field* (or *the torque distribution*) $\vec{c}_{\mathcal{S}'\to\mathcal{S}}(t)$ at a current point $A \in pos_{R_0}(\mathcal{S}_e, t)$.

   As in the case of a concentrated torque, we can use abbreviated notations whenever the context allows this: $\boxed{\vec{c}_{\mathcal{S}'\to\mathcal{S}}(A)}$, $\boxed{\vec{c}_{\to\mathcal{S}}(A)}$ ou $\boxed{\vec{c}(A)}$.

   Knowing the mass couple $\vec{c}$, it is possible to derive the volume torque (respectively, surface or line torque) using $\rho\vec{c}$ (respectively, $\rho_\Sigma\vec{c}$, $\rho_L\vec{c}$).

EXAMPLE OF THE NOTATION. Let us return to the example of the disc S rolling on a plane in Figure 3.1(a), and add here a concentrated torque that represents the resistance to the rolling exerted by the plane on the disc (Figure 3.2). The point of application of the torque is the point of contact A.

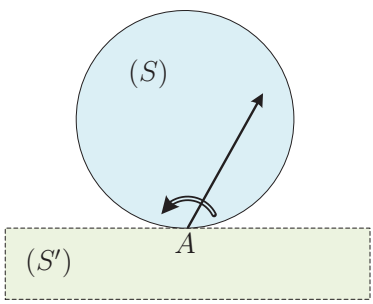


Figure 3.2. *Example of torque*

The contact torque exerted on the disc S can be denoted by $\vec{C}_{S'\to S}(A,t)$ or $\vec{C}_{\to S}(A,t)$, if we judge that there is no need to specify the support S'. Since the contact point A is common to both rigid bodies S and S', the abbreviated notation $\vec{C}(A,t)$ is not sufficient. ■

As with forces, a torque is applied on *the matter* (*one or more particles*) of a system and is not applied to a point (in space $\mathcal{E}$). When we say that a concentrated torque is applied to a point A, it must be understood that the torque is applied to a particle with position A.

Using the notations adopted earlier, for example $\vec{C}_{S'\to S}(A,t)$, we have chosen to highlight the *point of application* of the torque, rather than the particle on which the torque is exerted. These notations are sufficient to write equations in mechanics, for example to express the moment field due to a system of torques.

3.3. Efforts

Very often, forces and torques are involved concomitantly in a mechanics problem. Consider, for instance, two rigid bodies connected by a pin joint. The mechanical actions exerted by one body on the other due to the pin joint are represented by a force and a torque. When establishing the Lagrange's equations, we shall have to calculate the virtual power of both forces and torques alike. Thus, we are led to coin a new term that gathers "force" and "torque" together.

Definition. *Effort* is the generic term that designates force or torque. [3.1]

The term "effort" is, therefore, used when we want to indicate either a force or a torque without specifying which one.

REMARK. In the literature, the term "effort" is given another meaning as follows. By means of machines such as levers, we can use a smaller force to lift a heavy weight. A load is the weight that must be lifted or moved by the machines. An effort is defined as the force applied to the machines for lifting or carrying a load. This meaning is *not* used in this book. ■

We will use the symbol $\boxed{\mathcal{F}}$ to denote an effort or a set of efforts.

Let $\mathcal{S}$ be a system subjected to efforts exerted by another system $\mathcal{S}'$. These efforts may be at-a-distance or constraint efforts, either through direct contact or through an intermediate device such as a spring and a connecting rod. These efforts are generally made up of concentrated or distributed forces or torques. The set of efforts exerted by $\mathcal{S}'$ on $\mathcal{S}$ at an instant t is denoted by $\boxed{\mathcal{F}_{\mathcal{S}'\to\mathcal{S}}(t)}$, or, more simply, $\boxed{\mathcal{F}_{\mathcal{S}'\to\mathcal{S}}}$ if the time variable can be made implicit.

The notation $\boxed{\mathcal{F}_{\to\mathcal{S}}(t)}$ is used to designate "*the set of efforts (or the efforts system) applied to $\mathcal{S}$ at time t*", without specifying the origin of the efforts.

We have now categorized efforts based on their mode of distribution in space: concentrated efforts, mass efforts (or volume, surface or line efforts). This is a general classification. In the following section, we will categorize efforts based on two other, more specific, points of view: (i) efforts external to or internal to a system and (ii) given efforts or constraint efforts.

3.4. External and internal efforts

It has been seen that forces or torques can be classified into contact efforts or at-a-distance efforts, which can, themselves, be concentrated or distributed. Another possible classification is based on whether or not the agent of the effort is part of the studied system, and this leads us to distinguish between external and internal efforts.

3.4.1. *External effort*

Definition. [3.2]

1. An effort exerted on a system $\mathcal{S}$ is said to be *external to $\mathcal{S}$* if it is exerted by a system that is outside $\mathcal{S}$, that is, it has no material part that is common to $\mathcal{S}$.
2. *The external efforts exerted on a system $\mathcal{S}$* are the efforts exerted upon $\mathcal{S}$ by the material universe excepted for $\mathcal{S}$. The external efforts exerted on $\mathcal{S}$ at an instant t are denoted by $\boxed{\mathcal{F}_{ext\to\mathcal{S}}(t)}$.

The external efforts can be exerted on the whole of $\mathcal{S}$, as is the case in general for at-a-distance efforts such as gravity. They can also be exerted on one or more subsystems of $\mathcal{S}$ and such a subsystem may be a finite union of particles of $\mathcal{S}$, or a continuous subsystem occupying a volume, a surface or a line (for example, the contact forces exerted on part of the boundary of $\mathcal{S}$).

3.4.2. *Internal effort*

Definition. An effort is said be *internal* to a system $\mathcal{S}$, if it is exerted by a subsystem $\mathcal{S}_1$ of $\mathcal{S}$ on another subsystem $\mathcal{S}_2$ of $\mathcal{S}$, exterior to $\mathcal{S}_1$. [3.3]

The set of efforts internal to a system $\mathcal{S}$ at an instant t is denoted by $\boxed{\mathcal{F}_{int\to\mathcal{S}}(t)}$. This set is not always easy to determine as we will now see.

1. Let us assume that $\mathcal{S}$ is formed of a finite number of unconnected bodies $\mathcal{S}_i$, $i = 1, 2, \ldots$. The efforts $\mathcal{F}_{\mathcal{S}_i\to\mathcal{S}_j}$ exerted by a body $\mathcal{S}_i$ on another body $\mathcal{S}_j$, or the efforts $\mathcal{F}_{\mathcal{S}_j\to\mathcal{S}_i}$ exerted by $\mathcal{S}_j$ on $\mathcal{S}_i$, are efforts internal to $\mathcal{S}$. In this regard, we introduce a new terminology:

Definition. Let us consider two subsystems $\mathcal{S}_1$ and $\mathcal{S}_2$ of a system $\mathcal{S}$. [3.4]

The union of the efforts $\mathcal{F}_{\mathcal{S}_1\to\mathcal{S}_2}$ exerted by $\mathcal{S}_1$ on $\mathcal{S}_2$ and $\mathcal{F}_{\mathcal{S}_2\to\mathcal{S}_1}$ exerted by $\mathcal{S}_2$ on $\mathcal{S}_1$ at an instant t is called *the inter-efforts between $\mathcal{S}_1$ and $\mathcal{S}_2$ at t*. It is denoted by $\boxed{\mathcal{F}_{\mathcal{S}_1\leftrightarrow\mathcal{S}_2}(t)}$.

In particular, if $\mathcal{S}$ is a discrete system, that is a finite union of particles, the set of internal efforts is simple: this is the set of forces mutually exerted between the particles of the system.

2. Let us assume that $\mathcal{S}$ contains certain continuous bodies, that is bodies that are not reduced to a particle. In this case, the efforts inside each continuous body count among efforts internal to $\mathcal{S}$. These are not the classical efforts that are known in the mechanics of rigid bodies. Playing an important role in the mechanics of deformable bodies, they are not representable by forces or torques but are represented by a more sophisticated entity called the stress tensor. Fortunately, as will be seen in [5.2], the efforts internal to each body play *no* role in rigid bodies and *they can be ignored*.

To conclude, the set of efforts $\mathcal{F}_{int\to\mathcal{S}}$ internal to $\mathcal{S}$ includes the inter-efforts between the bodies in $\mathcal{S}$ *and* the efforts internal to each of these bodies. However, as the latter can be discarded in the framework of the mechanics of rigid bodies, the internal efforts $\mathcal{F}_{\text{int}\to\mathcal{S}}$ to be taken into account are reduced to the *inter-efforts* between the rigid bodies of the system.

• The set of efforts $\mathcal{F}_{\to\mathcal{S}}(t)$ applied to $\mathcal{S}$ includes all the efforts, internal and external, applied to $\mathcal{S}$. We thus have the following partition of the set of efforts $\mathcal{F}_{\to\mathcal{S}}$ exerted on the system $\mathcal{S}$:

$$\boxed{\mathcal{F}_{\to\mathcal{S}} = \mathcal{F}_{\text{ext}\to\mathcal{S}} \cup \mathcal{F}_{\text{int}\to\mathcal{S}}} \qquad \boxed{\emptyset = \mathcal{F}_{\text{ext}\to\mathcal{S}} \cap \mathcal{F}_{\text{int}\to\mathcal{S}}} \qquad [3.5]$$

3.5. Given efforts and constraint efforts

We now present another manner – different from [3.5] – to divide the set of efforts on $\mathcal{S}$, which is more commonly used in analytical mechanics. This consists of distinguishing between the given efforts and the constraint efforts.

Definition. An effort is a *given* effort if it is, *a priori*, a function of $(q_i)_{1\leq i\leq N}$, $(\dot{q}_i)_{1\leq i\leq N}$ and t, which are considered to be independent variables. [3.6]

EXAMPLES.

– The weight exerted on any system is a given force.

– Consider a rigid body belonging to a given system and a linear spring connecting the rigid body to a fixed support external to the system (respectively, another rigid body belonging to the same system). The force exerted by the spring is a given force as it is a function of the position of the rigid body (respectively, a function of the (relative) positions of the two rigid bodies).

– Similarly, the torque exerted by a spiral spring is a given effort. ■

Definition. [3.7]

A constraint effort is an effort caused by a mechanical joint. The mechanical joint may be

– an internal connection between two rigid bodies in the same system $\mathcal{S}$;

– or an external connection between a rigid body in the system $\mathcal{S}$ and the exterior.

Although the efforts due to springs are both constraint efforts and given efforts, let us adopt the following convention to simplify the discussion:

Convention. It is agreed that *efforts due to springs are considered as given efforts* but not as constraint efforts. [3.8]

This allows us to state the following assumption, which is verified in practice:

Assumption. [3.9]

– Constraint efforts are *not* given efforts.

– All efforts that are exerted on the rigid bodies of a system, other than constraint efforts, are *given*.

Constraint efforts are, therefore, *unknown quantities* of the problem akin to the motion of the system.

It is, thus, possible to partition the set of efforts applied on system $\mathcal{S}$ into given efforts and constraint efforts:

$$\boxed{\mathcal{F}_{\to\mathcal{S}} = \mathcal{F}_{\text{given}\to\mathcal{S}} \cup \mathcal{F}_{\text{constraint}\to\mathcal{S}}} \qquad \boxed{\emptyset = \mathcal{F}_{\text{given}\to\mathcal{S}} \cup \mathcal{F}_{\text{constraint}\to\mathcal{S}}} \tag{3.10}$$

As in the previous section, this partition does *not* include the internal efforts that exist within each rigid body.

It is seen that constraint efforts often arise from the contact between rigid bodies, while given efforts are at-a-distance efforts.

Gathering [3.5] and [3.10] leads to the following double equality:

$$\boxed{\mathcal{F}_{\to\mathcal{S}} = \mathcal{F}_{\text{ext}\to\mathcal{S}} \cup \mathcal{F}_{\text{int}\to\mathcal{S}} = \mathcal{F}_{\text{given}\to\mathcal{S}} \cup \mathcal{F}_{\text{constraint}\to\mathcal{S}}} \tag{3.11}$$

3.6. Moment field

Let us consider a system $\mathcal{S}$ which is, in general, made up of a finite union of particles and continuous subsystems. The efforts $\mathcal{F}(t)$ considered on $\mathcal{S}$ will always be made up of the following forces and torques:

- a finite number of concentrated forces: $\vec{F}_i(t) \equiv \vec{F}(B_i, t)$, applied at the point B_i, $i = 1, \ldots, I$,
- a mass force $\vec{f}(t)$, defined over a subset $\mathcal{S}_e$ of $\mathcal{S}$,
- a finite number of concentrated torques: $\vec{C}_j(t) \equiv \vec{C}(B'_j, t)$, applied at the point B'_j, $j = 1, \ldots, J$,
- mass torque $\vec{c}(t)$, defined over a subset $\mathcal{S}'_e$ of $\mathcal{S}$.

Definition and theorem.

Let us reason at a fixed instant t and consider the field $\mathcal{M}(t)$ defined by

$$\mathcal{M}(t) : \mathcal{E} \to E$$

$$A \mapsto \boxed{\begin{aligned} \vec{\mathcal{M}}(A,t) = & \sum_i \overrightarrow{AB_i} \times \vec{F}_i(t) + \int_{B \in pos_{R_0}(\mathcal{S}_e,t)} \overrightarrow{AB} \times \vec{f}(B,t)\mathrm{d}m \\ & + \sum_j \vec{C}_j(t) \qquad + \int_{pos_{R_0}(\mathcal{S}'_e,t)} \vec{c}(B,t)\mathrm{d}m \end{aligned}} \qquad [3.12]$$

where dm denotes the mass element surrounding the current integration point B; this is equal to $dm = \rho dV$, ρdS or $\rho d\ell$ depending on whether $\mathcal{S}$ occupies a volume, a surface or a line in space.

The above-defined mapping $\mathcal{M}(t)$ is called the *moment field* of the efforts $\mathcal{F}(t)$. We also say that the efforts $\mathcal{F}(t)$ *generate the moment field* $\mathcal{M}(t)$.

The vector $\vec{\mathcal{M}}_{\to\mathcal{S}}(A,t)$ is called *the moment of the efforts* $\mathcal{F}(t)$ about A at time t.

The *resultant force of the efforts* $\mathcal{F}(t)$*, evaluated at instant* t*,* is

$$\boxed{\vec{R}(t) = \sum_i \vec{F}_i(t) + \int_{pos_{R_0}(\mathcal{S}_e,t)} \vec{f}(A,t)\mathrm{d}m} \qquad [3.13]$$

PROOF. By writing [3.12] for two arbitrary points A and A' in $\mathcal{E}$ and subtracting the obtained equalities, we get

$$\vec{\mathcal{M}}_{\to\mathcal{S}}(A,t) - \vec{\mathcal{M}}_{\to\mathcal{S}}(A',t) = \overrightarrow{AA'} \times \left(\sum_i \vec{F}_i(t) + \int_{pos_{R_0}(\mathcal{S}_e,t)} \vec{f}(B,t)\mathrm{d}m \right) \qquad [3.14]$$

As a result, the moments about different points are related by

$$\vec{\mathcal{M}}_{\to\mathcal{S}}(A,t) = \vec{\mathcal{M}}_{\to\mathcal{S}}(A',t) + \overrightarrow{AA'} \times \vec{R}(t)$$

where vector $\vec{R}(t)$ is defined by [3.13], after relabeling the integration variable B in the integrand of [3.14] into A. ■

• The mass force $\vec{f}(t)$ (respectively, the mass torque $\vec{c}(t)$) may be defined over a volume, a surface, a line or a union of these entities. The integrals in [3.12] and [3.13] may thus be sums of the volume, surface or line integrals.

• We assumed mass efforts for the sake of definiteness. However, it is indeed possible to consider other types of distribution of efforts. If, for instance, there is a volume (or a surface or line) force, all we need to do is add to the first integral in [3.12] a similar integral with dV (or, respectively, dS or $d\ell$) instead of dm. Once again, the integrals in [3.12] and [3.13] may be sums of volume, surface or line integrals. In the general case when concentrated, mass, volumetric, surface and line forces coexist, the resultant force of the efforts, for instance, can be written using the evident notations:

$$\vec{R}(t) = \sum_i \vec{F}_i(t) + \int_{pos_{R_0}(\mathcal{S}_e,t)} \vec{f}^m(A,t)\mathrm{d}m + \int_V \vec{f}^v(A,t)\mathrm{d}V + \int_S \vec{f}^s(A,t)\mathrm{d}S + \int_L \vec{f}^\ell(A,t)\mathrm{d}\ell$$

The integral is a volume integral over V, a surface integral over S or a curvilinear integral along L.

• The moments of the efforts about different points are related through the relationship:

$$\forall A, B \in \mathcal{E}, \boxed{\vec{\mathcal{M}}(B,t) = \vec{\mathcal{M}}(A,t) + \overrightarrow{BA} \times \vec{R}(t)} \qquad [3.15]$$

The moment field $\mathcal{M}(t)$ is entirely defined by the resultant force $\vec{R}(t)$ and the moment of the efforts $\vec{\mathcal{M}}(A,t)$ about a point A, i.e. by the six scalar components:

$$3 \text{ for the resultant force } \left\{\begin{array}{c} R_x \\ R_y \\ R_z \end{array}\right\} \text{ or } \left\{\begin{array}{c} X \\ Y \\ Z \end{array}\right\} \text{ and 3 for the moment } \left\{\begin{array}{c} \mathcal{M}_x(A) \\ \mathcal{M}_y(A) \\ \mathcal{M}_z(A) \end{array}\right\} \text{ or } \left\{\begin{array}{c} L(A) \\ M(A) \\ N(A) \end{array}\right\}$$

The moment field $\mathcal{M}(t)$ may be written in the following equivalent form:

$$\boxed{\mathcal{M}(t) = \left[\begin{array}{c} \vec{R}(t) \\ \vec{\mathcal{M}}(A,t) \end{array}\right]_A} \qquad [3.16]$$

• In a similar way to what was done for efforts, we write:

- $\boxed{\mathcal{M}_{\mathcal{S}'\to\mathcal{S}}(t)}$ the moment field exerted by a system $\mathcal{S}'$ on another system $\mathcal{S}$ at instant t.
- $\boxed{\mathcal{M}_{ext\to\mathcal{S}}(t)}$ the moment field of all external efforts exerted on $\mathcal{S}$ at instant t.
- $\boxed{\mathcal{M}_{int\to\mathcal{S}}(t)}$ the moment field of all efforts internal to $\mathcal{S}$ at instant t.
- $\boxed{\mathcal{M}_{\mathcal{S}_1\leftrightarrow\mathcal{S}_2}(t)}$ the moment field of the inter-efforts between two systems $\mathcal{S}_1$ and $\mathcal{S}_2$ at t.
- Finally, when there is no need to specify the origin of the efforts, we write $\boxed{\mathcal{M}_{\to\mathcal{S}}(t)}$, which designates the moment field of external as well as internal efforts applied to $\mathcal{S}$.

• Let us recall a well-known result in mechanics:

Action and reaction theorem (or the principle of mutual actions). Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two disjoint systems (that is, they have no material part in common). At any instant, the moment field of the efforts exerted by $\mathcal{S}_1$ on $\mathcal{S}_2$ and the moment field of the efforts exerted by $\mathcal{S}_2$ on $\mathcal{S}_1$ are opposite:

$$\forall t, \boxed{\mathcal{M}_{\mathcal{S}_1\to\mathcal{S}_2}(t) = -\mathcal{M}_{\mathcal{S}_2\to\mathcal{S}_1}(t)} \qquad [3.17a]$$

The above relationship is an *equality between moment fields*, and is equivalent to equalities between the resultant force and the moments about any point:

$$\forall t, \qquad \boxed{\begin{array}{l} .\ \vec{R}_{\mathcal{S}_1\to\mathcal{S}_2}(t) = -\vec{R}_{\mathcal{S}_2\to\mathcal{S}_1}(t) \\ .\ \forall A, \quad \vec{\mathcal{M}}_{\mathcal{S}_1\to\mathcal{S}_2}(A,t) = -\vec{\mathcal{M}}_{\mathcal{S}_2\to\mathcal{S}_1}(A,t) \end{array}} \qquad [3.17b]$$

In other words, the sum of the moment fields of the inter-efforts between the rigid bodies in $\mathcal{S}$ is zero:

$$\forall t, \boxed{\mathcal{M}_{\mathcal{S}_1\leftrightarrow\mathcal{S}_2}(t) = \mathcal{M}_{\mathcal{S}_1\to\mathcal{S}_2}(t) + \mathcal{M}_{\mathcal{S}_2\to\mathcal{S}_1}(t) = 0} \qquad [3.18]$$

4

Virtual Kinematics

In this chapter, we will introduce different concepts related to the so-called "virtual kinematics": the virtual velocity of a particle, the virtual velocity field of a rigid body or a system, virtual angular velocity as well as the composition of virtual velocities.

Virtual kinematics is constructed on the model of real kinematics, as can be observed by comparing this chapter with Chapters 1 and 2. However, the term "*virtual*" is used as a reminder; although a virtual quantity is analogous to a real quantity, it can take arbitrary values that have nothing to do with the real motion of the mechanical system being studied. The concept of virtual velocity, which is not a physical entity, is the basic ingredient of analytical mechanics. This concept makes it possible to define the "virtual power", which we will see in Chapter 5 and which comes into play in the principle of virtual powers (PVP) [5.1]. The PVP in turn enables us to establish the Lagrange's equations.

In the following, use will be made of the parameterization [2.19] to define the position of any system with respect to the common reference frame R_0. We use R_1 to designate an arbitrary reference frame. As in Chapter 2, it is assumed that in the most general case, the rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 in R_0 depends on the position parameters q and t, which is expressed by [2.23]: $\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{01}(q,t)$.

A virtual quantity will be represented by a symbol that is the same as the symbol for its real counterpart, but supplemented with an asterisk $(*)$.

4.1. Virtual derivative of a vector with respect to a reference frame

Consider a vector quantity (e.g. the position vector of a particle) whose observation result with respect to the common reference frame R_0 is a vector $\vec{W} \in E$ that is a function of q, t (in R_0 we write $\vec{W}$ rather than $\vec{W}^{(0)}$). The time derivative, with respect to a reference frame R_1, of the vector $\vec{W}$ is defined by [1.38]. Taking into account the dependence with respect to (q,t), the derivative is written as

$$\frac{d_{R_1}\vec{W}}{dt} = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)\dot{q}_i + \frac{\partial_{R_1}\vec{W}}{\partial t} \qquad [4.1]$$

Based on the above model, we introduce the *virtual derivative of the vector* $\vec{W}$ *with respect to a given reference frame* R_1:

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

Definition. *The virtual derivative, with respect to a reference frame R_1, of a vector $\vec{W}$*, denoted by $\boxed{\dfrac{d^*_{R_1}\vec{W}}{dt} \text{ or in abbreviated form} \dfrac{d^*_1\vec{W}}{dt}}$, is defined as

$$\boxed{\frac{d^*_{R_1}\vec{W}}{dt} \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)\dot{q}_i^*} \in E, \qquad [4.2]$$

where $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ is an *arbitrary* n-tuple of $\mathbb{R}^n$.

The virtual derivative [4.2] is defined in a manner that is similar to the definition for the real derivative [4.1]:

– we cancel the term $\frac{\partial_{R_1}\vec{W}}{\partial t}$ in [4.1],

– and we replace the $\dot{q}_i$ with any scalars, denoted by $\dot{q}_i^*$ by analogy. However, despite the notation, $\dot{q}_i^*$ is *not* the time derivative of a certain function $q_i^*(t)$ – it is just an *arbitrary* quantity.

The virtual derivative in [4.2] is associated with the parameterization [2.19]. However, for the sake of brevity, this will not be repeated at every instance.

The flowchart in calculating the virtual derivative is the same as in [1.39].

If $\lambda(q,t)$ is a scalar function of q, t, the virtual derivative of λ is defined in a similar manner:

Definition. *The virtual derivative of λ*, denoted by $\boxed{\dfrac{d^*\lambda}{dt}}$, is, by definition:

$$\boxed{\frac{d^*\lambda}{dt} \equiv \sum_{i=1}^{n}\frac{\partial\lambda}{\partial q_i}\dot{q}_i^*} \qquad [4.3]$$

Theorem.

$$\frac{d^*_{R_1}\vec{W}}{dt} = \vec{0},\ \forall\dot{q}_i^* \quad \Leftrightarrow \quad \bar{\bar{Q}}_{10}.\vec{W} \text{ possibly depends on } t, \text{but not on } q. \qquad [4.4]$$

A particular case where the right-hand side of the equivalence is true occurs when $\vec{W}$ is *fixed in* R_1 (see definition [1.35]).

PROOF.

$$\frac{d^*_{R_1}\vec{W}}{dt} = \vec{0},\ \forall\dot{q}_i^* \Leftrightarrow \sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)\dot{q}_i^* = \vec{0},\ \forall\dot{q}_i^* \quad \text{according to definition [4.2]}$$
$$\Leftrightarrow \forall i, \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right) = 0 \qquad \blacksquare$$

Theorem. $\forall$ reference frame $R_1, \forall \vec{U}, \vec{V}, \vec{W} \in E, \forall \lambda \in \mathbb{R}$,

$$\frac{d^*_{R_1}}{dt}(\vec{U}+\vec{V}) = \frac{d^*_{R_1}\vec{U}}{dt} + \frac{d^*_{R_1}\vec{V}}{dt} \quad [4.5]$$

$$\frac{d^*_{R_1}}{dt}(\lambda\vec{W}) = \frac{d^*\lambda}{dt}\,\vec{W} + \lambda\frac{d^*_{R_1}\vec{W}}{dt} \quad [4.6]$$

$$\frac{d^*}{dt}(\vec{U}.\vec{V}) = \frac{d^*_{R_1}\vec{U}}{dt}.\vec{V} + \vec{U}.\frac{d^*_{R_1}\vec{V}}{dt} \quad [4.7]$$

Note that the index R_1 of the reference frame appears on the right-hand side but not on the left-hand side of [4.7]. Indeed, as $\vec{U}.\vec{V}$ is a scalar, its virtual derivative, defined by [4.3], does not depend on any reference frame.

PROOF.

– As equality [4.5] is straightforward, let us prove [4.6]:

$$\frac{d^*_{R_1}}{dt}(\lambda\vec{W})\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\lambda\vec{W}\right)\dot{q}_i^*$$

where $\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\lambda\vec{W}\right) = \frac{\partial\lambda}{\partial q_i}\bar{\bar{Q}}_{10}.\vec{W} + \lambda\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)$. Hence, using $\bar{\bar{Q}}_{01}.\bar{\bar{Q}}_{10} = \bar{\bar{I}}$:

$$\frac{d^*_{R_1}}{dt}(\lambda\vec{W})\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\lambda\vec{W}\right)\dot{q}_i^* = \sum_{i=1}^{n}\frac{\partial\lambda}{\partial q_i}\dot{q}_i^*\,\vec{W} + \lambda\,\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)\dot{q}_i^*$$

– let us now prove [4.7]:

$$\begin{aligned}
\frac{d^*}{dt}(\vec{U}.\vec{V}) &= \sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\vec{U}.\vec{V}\right)\dot{q}_i^* \text{ according to definition [4.3]}\\
&= \sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{U}.\,\bar{\bar{Q}}_{10}.\vec{V}\right)\dot{q}_i^* \text{ knowing that } \vec{U}.\vec{V} = \left(\bar{\bar{Q}}_{10}.\vec{U}\right).\left(\bar{\bar{Q}}_{10}.\vec{V}\right)\\
&= \sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{U}\right)\dot{q}_i^*.\left(\bar{\bar{Q}}_{10}.\vec{V}\right) + \left(\bar{\bar{Q}}_{10}.\vec{U}\right).\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{V}\right)\dot{q}_i^*\\
&= \vec{V}.\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{U}\right)\dot{q}_i^* + \vec{U}.\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{V}\right)\dot{q}_i^*\\
&\qquad \text{using } \vec{a}.\left(\bar{\bar{Q}}_{10}.\vec{b}\right) = \left(\bar{\bar{Q}}_{01}.\vec{a}\right).\vec{b}
\end{aligned}$$

Note that all derivatives on the right-hand sides of the above equalities are standard derivatives in E. ■

Theorem. Let $\vec{W} = \alpha\vec{x}_1 + \beta\vec{y}_1 + \gamma\vec{z}_1 \in E$, where $b_1 = (\vec{x}_1, \vec{y}_1, \vec{z}_1)$ is a vector basis of E, *fixed in* R_1. Then

$$\boxed{\frac{d^*_{R_1}\vec{W}}{dt} = \frac{d^*\alpha}{dt}\vec{x}_1 + \frac{d^*\beta}{dt}\vec{y}_1 + \frac{d^*\gamma}{dt}\vec{z}_1} \quad [4.8]$$

PROOF. Apply relationships [4.5] and [4.6]. ■

Dependence of the virtual derivative with respect to the reference frame

Definition [4.2] clearly shows that the virtual derivative $\dfrac{d^*_{R_1}\vec{W}}{dt}$ with respect to a reference frame R_1 depends *a priori* on R_1, which justifies the index R_1 in the notation. Although the multiplication of the vectors $\dfrac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)$ by the arbitrary $\dot{q}_i^*$ means that the *value* of the virtual derivative becomes arbitrary and independent of the reference frame, *the analytical expression* [4.2] of the virtual derivative does indeed depend on the reference frame R_1. This point will be illustrated by the examples in later sections.

However, under hypothesis [2.26], which was introduced for real velocities, we have the following property of independence:

Theorem.

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

Then, the virtual derivative [4.2] does *not* depend on R_1 and is written *without* the reference frame index:

$$\boxed{\frac{d^*\vec{W}}{dt} = \sum_{i=1}^{n} \frac{\partial \vec{W}}{\partial q_i}\dot{q}_i^*} \qquad [4.9]$$

PROOF. Straightforward. If $\bar{\bar{Q}}_{01}$ does *not* depend on q, then this is the same for $\bar{\bar{Q}}_{10} = \bar{\bar{Q}}_{01}^{-1}$ and it can be taken out of the derivative in [4.2]:

$$\frac{d^*_{R_1}\vec{W}}{dt} \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)\dot{q}_i^* = \underbrace{\bar{\bar{Q}}_{01}.\bar{\bar{Q}}_{10}}_{=\bar{\bar{I}}}.\sum_{i=1}^{n}\frac{\partial \vec{W}}{\partial q_i}\dot{q}_i^* \qquad \blacksquare$$

4.2. Virtual velocity of a particle

Definition. *The virtual velocity (VV) of the particle p, with respect to R_1 and at an instant t, associated with* (or *resulting from*) *the parameterization* [2.19], denoted by $\vec{V}^*_{R_1}(p)$, is defined as

$$\boxed{\vec{V}^*_{R_1}(p) \equiv \frac{d^*_{R_1}\overrightarrow{O_1P}}{dt} \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)\dot{q}_i^*} \qquad [4.10]$$

where O_1 is a fixed point in R_1, the position vector $\overrightarrow{O_1P}(q,t)$ is given by [2.21] and $(\dot{q}_1^*,\ldots,\dot{q}_n^*)$ is an *arbitrary* n-tuple of $\mathbb{R}^n$.

The virtual velocity is associated with the parameterization [2.19], but this will not be repeated systematically, for brevity.

REMARK. In definition [4.10], $(\dot{q}_1^*,\ldots,\dot{q}_n^*)$ is an arbitrary n-tuple in $\mathbb{R}^n$.

– To emphasize the arbitrary nature of this n-tuple, we can say that [4.10] gives (the expression of) *the most general VV* of the particle p or, more briefly, *the VV*.

– If we work with a given n-tuple $(\dot{q}_1^*,\ldots,\dot{q}_n^*)$, we will talk of *a VV*.

If we are looking at values, we can talk about the *virtual velocities* or *a virtual velocity*:

– when $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ is given all the values in $\mathbb{R}^n$, this generates the *set of all virtual velocities*,

– each value taken by the n-tuple $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ corresponds to *one virtual velocity* of the particle p.

In this book, we will work with one of the above meanings, but we will not always distinguish between them in a systematic manner. We will use "the virtual velocity" or "a virtual velocity" interchangeably, knowing that the specific meaning is provided by the context. ■

Using assumption [2.26], which was introduced for the real velocities, we can immediately transform [4.10] into another, simpler expression:

Theorem.

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

Then

$$\boxed{\vec{V}_{R_1}^*(p) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1P}}{\partial q_i} \dot{q}_i^*} \qquad [4.11]$$

Analogy between real velocity and virtual velocity

The term *virtual velocity* is used to highlight the analogy with real velocity:

[4.10] is analogous to [2.25]: $\vec{V}_{R_1}(p,t) = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}$

[4.11] is analogous to [2.27]: $\vec{V}_{R_1}(p,t) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1P}}{\partial q_i}\dot{q}_i + \frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}$

However, apart from this analogy, *the VV has absolutely no connection with the real velocity*. The velocity of a particle p with respect to a reference frame R_1 at an instant t is a well-defined vector $\vec{V}_{R_1}(p,t)$, while the VV $\vec{V}_{R_1}^*(p)$ may be an arbitrary vector corresponding to an arbitrary n-tuple $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$.

The real velocity is not always a particular virtual velocity. The real velocity is an element of the set of virtual velocities only if $\frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t} = \bar{\bar{Q}}_{01}.\frac{\partial \overrightarrow{OP^{(1)}}}{\partial t} = \vec{0}$, i.e., if $\frac{\partial \overrightarrow{OP^{(1)}}}{\partial t} = \vec{0}$.

With regard to the physical units, if the $\dot{q}_i^*$ have the same units as the $\dot{q}_i$, then the virtual velocity has the same unit as a real velocity.

Moving from the real velocity to the virtual velocity

With the help of the above-mentioned analogy, it can be seen how to obtain the VV when we know the analytical expression for the real velocity: [4.12]

– from the expression for the real velocity, we remove the term $\frac{\partial_{R_1}\overrightarrow{O_1P}}{\partial t}$, that is, we remove all terms that are not coefficients of a $\dot{q}_i$,

– and we replace the $\dot{q}_i$ by arbitrary scalars, which have been denoted by $\dot{q}_i^*$.

Example and counter-example

Theorem [4.11] requires the hypothesis that $\bar{\bar{Q}}_{01}$ depends only on time, not on q. This hypothesis is often verified in practice; however, attention should be drawn on the fact that there are some cases where it is not verified. In order to see the importance of this hypothesis, let us return to the example presented in section 2.6 and calculate the virtual velocities $\vec{V}^*_{R_1}(a)$ and $\vec{V}^*_{R_0}(a)$ for the particle a, using two different parameterizations, *ibid*, of which one verifies hypothesis [2.26] and the other does not.

Recall that for both parameterizations considered, the primitive parameters of the problem are the same: the four coordinates X, Y, x, y and the two angles φ, θ.

1. First parameterization: The constraint equation $\varphi = \omega t$ is classified as a complementary equation, with the result that the retained parameters are $q = (X, Y, x, y, \varphi, \theta)$. The position of the particle a in R_0 and at instant t is $A = A(X, Y, x, y, \varphi)$.

   (a) Using relationship [4.10] with the point O_1, which is fixed in R_1, being taken equal to O and $\overrightarrow{O_1A} = x\vec{x}_1(\varphi) + y\vec{y}_1(\varphi)$, we obtain the virtual velocity with respect to R_1 for the particle a:

   $$\vec{V}^*_{R_1}(a) = \dot{x}^*\vec{x}_1 + \dot{y}^*\vec{y}_1 \qquad [4.13]$$

   The same result is obtained if we start from the real velocity [2.29] and transform this into the virtual velocity using the procedure described earlier.

   On the other hand, as the rotation tensor $\bar{\bar{Q}}_{01}$ depends on φ, hypothesis [2.26] is not satisfied and we *cannot* use [4.11]. Indeed, relationship [4.11] gives

   $$\vec{V}^*_{R_1}(a) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1A}}{\partial q_i}\dot{q}_i^* = \underbrace{\frac{\partial \overrightarrow{O_1A}}{\partial x}}_{\vec{x}_1}\dot{x}^* + \underbrace{\frac{\partial \overrightarrow{O_1A}}{\partial y}}_{\vec{y}_1}\dot{y}^* + \underbrace{\frac{\partial \overrightarrow{O_1A}}{\partial \varphi}}_{x\vec{y}_1 - y\vec{x}_1}\dot{\varphi}^*$$
   $$= (\dot{x}^* - y\dot{\varphi}^*)\vec{x}_1 + (\dot{y}^* + x\dot{\varphi}^*)\vec{y}_1 : \text{which is false.}$$

   (b) For the sake of comparison, let us calculate the virtual velocity with respect to R_0:

   $$\vec{V}^*_{R_0}(a) \equiv \bar{\bar{Q}}_{00}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{00}.\overrightarrow{OA}\right)\dot{q}_i^* \underset{\bar{\bar{Q}}_{00}=\bar{\bar{I}}}{=} \sum_{i=1}^{n} \frac{\partial \overrightarrow{OA}}{\partial q_i}\dot{q}_i^* \qquad [4.14]$$

   where $\overrightarrow{OA} = X\vec{e}_1 + Y\vec{e}_2 + x\vec{x}_1(\varphi) + y\vec{y}_1(\varphi)$. We obtain

   $$\vec{V}^*_{R_0}(a) = \dot{X}^*\vec{e}_1 + \dot{Y}^*\vec{e}_2 + (\dot{x}^* - y\dot{\varphi}^*)\vec{x}_1 + (\dot{y}^* + x\dot{\varphi}^*)\vec{y}_1. \qquad [4.15]$$

2. Second parameterization: In this parameterization, we decide to classify $\varphi = \omega t$ as a primitive and not as a complementary equation and we will see the differences that result from this. The retained parameters are, thus, $q = (X, Y, x, y, \theta)$ and t, and the dependence on t occurs via $\varphi = \omega t$. The position of the particle a, in R_0 and at instant t, is $A = A(X, Y, x, y, t)$.

   (a) Relationship [4.10], this time with $\overrightarrow{O_1A} = x\vec{x}_1(t) + y\vec{y}_1(t)$, and using the same calculation as in the first parameterization, leads to

   $$\vec{V}^*_{R_1}(a) = \dot{x}^*\vec{x}_1 + \dot{y}^*\vec{y}_1 \qquad [4.16]$$

   which is identical to [4.13], provided that, of course, one makes $\vec{x}_1 = \vec{x}_1(\varphi(t))$, $\vec{y}_1 = \vec{y}_1(\varphi(t))$ in [4.13], and $\vec{x}_1 = \vec{x}_1(t)$, $\vec{y}_1 = \vec{y}_1(t)$ in [4.16].

As the rotation tensor $\bar{\bar{Q}}_{01}$ does not depend on q here, we can use [4.11]:

$$\vec{V}^*_{R_1}(a) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1A}}{\partial q_i}\dot{q}_i = \underbrace{\frac{\partial \overrightarrow{O_1A}}{\partial x}}_{\vec{x}_1}\dot{x}^* + \underbrace{\frac{\partial \overrightarrow{O_1A}}{\partial y}}_{\vec{y}_1}\dot{y}^*$$

We thus arrive once again at [4.16].

(b) The calculation of the virtual velocity with respect to R_0 is carried out in the same way. Relationship [4.14], this time with $\overrightarrow{OA} = X\vec{e}_1 + Y\vec{e}_2 + x\vec{x}_1(t) + y\vec{y}_1(t)$, leads to

$$\vec{V}^*_{R_0}(a) = \dot{X}^*\vec{e}_1 + \dot{Y}^*\vec{e}_2 + \dot{x}^*\vec{x}_1 + \dot{y}^*\vec{y}_1 \qquad [4.17]$$

This expression is different from [4.15], which proves that the virtual velocity depends, *a priori*, on the chosen parameterization. ■

Dependence of the virtual velocity on time

According to [4.10], the VV of a particle depends, *a priori*, on t via $\overrightarrow{O_1P}(q(t),t)$ and $\bar{\bar{Q}}_{01}(q(t),t) = \bar{\bar{Q}}^{-1}_{10}(q(t),t)$, such that, strictly speaking, it should be denoted by $\vec{V}^*_{R_1}(p,t)$. Expression [4.11] leads to the same observation: it can be seen that the VV of a particle depends, *a priori*, on t via $\overrightarrow{O_1P}(q(t),t)$ and we should once again write $\vec{V}^*_{R_1}(p,t)$. We can, further, verify the dependence on t through the above example.

That being said, however, the VV is *not*, in general, a continuous time function as it is the product of functions of $\bar{\bar{Q}}_{01}$ and $\overrightarrow{O_1P}$, which are continuous in time, with the arbitrary quantities $\dot{q}^*_i$, which may be discontinuous in time.

This discontinuity in time does not pose any problems for the theory. As will be seen in Chapter 6, in order to establish the Lagrange's equations, the $\dot{q}^*_i$ must be arbitrary *at every instant t*, but there is no need for it to be continuous in time, unlike the real $\dot{q}_i$. The virtual velocity field (VVF) at one instant t may have no continuity with the VVF at the next instant.

To summarize, even if the VV does indeed depend on t, this dependence is discontinuous and has no effect on the theory. This is why we decide to *make implicit the argument t in* $\vec{V}^*_{R_1}(p,t)$ to simply write $\vec{V}^*_{R_1}(p)$.

Dependence of the virtual velocity on the reference frame

The expressions [4.13]–[4.17] obtained in the above example clearly show that the virtual velocity depends, *a priori*, on the reference frame with respect to which it is calculated. This justifies the index R_1 in the notation $\vec{V}^*_{R_1}(p)$ in relationships [4.10] and [4.11].

One should distinguish between the expression $\vec{V}^*_{R_1}(p)$ and its value:

1. *The expression* [4.10] for $\vec{V}^*_{R_1}(p)$ does indeed depend on the reference frame R_1 (compare, e.g., [4.13] and [4.15]).

2. Regarding the *value* of $\vec{V}^*_{R_1}(p)$, the case is different. As the $\dot{q}^*_i$ in [4.10] are arbitrary, when the $\dot{q}^*_i$ are given all the values in the set of real numbers $\mathbb{R}$, the virtual velocity vector spans the whole vector space E. Thus, for example:

   – when $\dot{x}^*, \dot{y}^*$ in [4.13] take all the values in $\mathbb{R}$, the vector $\vec{V}^*_{R_1}(a)$ takes all the values in E,

   – when $\dot{X}^*, \dot{Y}^*, \dot{x}^*, \dot{y}^*, \dot{\varphi}^*$ in [4.15] take all values in $\mathbb{R}$, the vector $\vec{V}^*_{R_0}(a)$ also takes all values in E.

As we always obtain the same set E, regardless of the reference frame being considered, we say that *the value* of $\vec{V}^*_{R_1}(p)$ is independent of the reference frame R_1.

It is important to retain the index R_1 in the notation $\vec{V}^*_{R_1}(p)$ for the virtual velocity. As will be seen later, this allows one to write the composition formula for virtual velocities similar to the formula for real velocities, for example [4.49]: $V^*_{R_1S} = V^*_{R_1S(R_2)} + V^*_{R_2S}$.

Below is a case when the virtual velocity does not depend on the reference frame with respect to which we calculate the virtual velocity:

Theorem. The virtual velocity $\vec{V}^*_{R_1}(p)$ has *the same expression* for all the reference frames R_1 that verify the hypothesis [2.33], which is a little stronger than [2.26]:

HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and the point O_1 fixed in R_1 do *not* depend on q.

The expression for the virtual velocity is similar to [4.11] but *no longer depends on* R_1. The virtual velocity will be written *without the reference frame index and without the point* O_1:

$$\boxed{\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{P}}{\partial q_i} \dot{q}_i^*} \qquad [4.18]$$

where $\dfrac{\partial \overrightarrow{P}}{\partial q_i}$ designates $\dfrac{\partial \overrightarrow{O'P}}{\partial q_i}$, O' being any point independent of q.

PROOF. As it is assumed that the rotation tensor $\bar{\bar{Q}}_{01}$ does not depend on q, the virtual velocity is given by [4.11]: $\vec{V}^*_{R_1}(p) = \sum_{i=1}^{n} \dfrac{\partial \overrightarrow{O_1P}}{\partial q_i} \dot{q}_i^*$.

Let us write $\overrightarrow{O_1P} = \overrightarrow{O'P} - \overrightarrow{O'O_1}$, where O' is any point other than O_1 and independent of q. We thus have $\dfrac{\partial \overrightarrow{O_1P}}{\partial q_i} = \dfrac{\partial \overrightarrow{O'P}}{\partial q_i} - \dfrac{\partial \overrightarrow{O'O_1}}{\partial q_i}$ where $\dfrac{\partial \overrightarrow{O'O_1}}{\partial q_i} = \vec{0}$ as the points O_1, O' do not depend on q. We can, thus, rewrite [4.11] in the form:

$$\vec{V}^*_{R_1}(p) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O'P}}{\partial q_i} \dot{q}_i^*$$

The virtual velocity in a reference frame that is different from R_1 has the same expression, as the vector $\overrightarrow{O'P}$ on the right-hand side remains the same. ■

EXAMPLE. In the example presented earlier, if we use the second parameterization assuming that the point O_1 is fixed in R_0, the hypothesis from [4.18] is verified. The coordinates X, Y are thus constant and the retained parameters are reduced to (x, y, θ) and $\overrightarrow{OA} = x\vec{x}_1(t) + y\vec{y}_1(t)$. In place of [4.17], we obtain: $\vec{V}^*_{R_0}(a) = \dot{x}^*\vec{x}_1 + \dot{y}^*\vec{y}_1$.

In comparison to [4.16], we see that

$$\vec{V}^*_{R_1}(a) = \vec{V}^*_{R_0}(a) = \dot{x}^*\vec{x}_1 + \dot{y}^*\vec{y}_1$$

The expression for the virtual velocity is the same in both reference frames R_1 and R_0. It depends on R_1 inasmuch as it contains the vectors $\vec{x}_1, \vec{y}_1$, but it does not depend on R_1 in the sense of the above theorem statement. In this instance, when we replace R_1 with R_0 we arrive at the same expression for virtual velocity. ■

4.3. Virtual angular velocity

We will bring into play a reference frame R_2 other than R_1 and as in section 2.7 it is assumed that the position of the rigid body $S(R_2)$ defined by R_2 depends on (q,t) (which is true in practice because, in the applications of the theory, R_2 is defined by a rigid body of the system studied).

Theorem and definition. Composite virtual derivative of a vector. Consider a vector quantity whose observation result with respect to the common reference R_0 is vector $\vec{W} \in E$, a function of q, t. We have:

$$\boxed{\frac{d_1^*\vec{W}}{dt} = \frac{d_2^*\vec{W}}{dt} + \vec{\Omega}_{12}^* \times \vec{W}} \quad \text{with} \quad \boxed{\vec{\Omega}_{12}^* \equiv \vec{\Omega}_{R_1R_2}^* \equiv \frac{1}{2}\sum_{j=1}^{3} \vec{b}_j \times \frac{d_{R_1}^*\vec{b}_j}{dt}}, \qquad [4.19]$$

where $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ is an orthonormal basis of E, *fixed in* R_2 (Figure 1.8. The vector $\vec{\Omega}_{12}^*$ is called *the virtual angular velocity of* R_2 *with respect to* R_1*(at instant* t *and associated with parameterization* [2.19]).

The *skew-symmetric* tensor $\bar{\bar{\Omega}}_{12}^*$ associated with $\vec{\Omega}_{12}^*$ is called *the virtual angular velocity tensor of* R_2 *with respect to* R_1 *(at* t *and associated with parameterization* [2.19]*)*. It is related to the rotation tensors $\bar{\bar{Q}}_{01}, \bar{\bar{Q}}_{02}$ and $\bar{\bar{Q}}_{12}$ through

$$\boxed{\bar{\bar{\Omega}}_{12}^* = \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n} \frac{\partial \bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}_i^*\right).\bar{\bar{Q}}_{20}} \qquad [4.20]$$

The virtual angular velocity is associated with parameterization [2.19], but this will not be repeated systematically, for brevity.

PROOF.

- Proof of [4.19]. By denoting $\vec{W} = \sum_{j=1}^{3} \beta_j \vec{b}_j$, we have, according to [4.5] and [4.6]

$$\frac{d_1^*\vec{W}}{dt} = \sum_{j=1}^{3} \frac{d_1^*}{dt}(\beta_j \vec{b}_j) = \sum_{j=1}^{3}\left(\frac{d^*\beta_j}{dt}\vec{b}_j + \beta_j \frac{d_1^*\vec{b}_j}{dt}\right)$$

The term $\sum_{j=1}^{3} \beta_j \frac{d_1^*\vec{b}_j}{dt}$ can be transformed as follows:

$$\sum_{j=1}^{3} \beta_j \frac{d_1^*\vec{b}_j}{dt} = \sum_{j=1}^{3}\left(\vec{W}.\vec{b}_j\right)\frac{d_1^*\vec{b}_j}{dt} = \vec{W} \times (\sum_{j=1}^{3} \frac{d_1^*\vec{b}_j}{dt} \times \vec{b}_j) + \sum_{j=1}^{3}\left(\vec{W}.\frac{d_1^*\vec{b}_j}{dt}\right)\vec{b}_j$$

where according to [4.7], $\vec{W}.\frac{d_1^*\vec{b}_j}{dt} = \frac{d^*(\vec{W}.\vec{b}_j)}{dt} - \vec{b}_j \frac{d_1^*\vec{W}}{dt}$. Hence

$$\sum_{j=1}^{3} \beta_j \frac{d_1^*\vec{b}_j}{dt} = \vec{W} \times \sum_{j=1}^{3}\left(\frac{d_1^*\vec{b}_j}{dt} \times \vec{b}_j\right) + \sum_{j=1}^{3} \frac{d^*\beta_j}{dt}\vec{b}_j - \sum_{j=1}^{3}\left(\vec{b}_j \frac{d_1^*\vec{W}}{dt}\right)\vec{b}_j$$

On the one hand, according to [4.8], $\sum_{j=1}^{3} \frac{d^*\beta_j}{dt}\vec{b}_j = \frac{d_2^*\vec{W}}{dt}$, on the other hand, as the basis $(\vec{b}_1, \vec{b}_2, \vec{b}_3)$ is orthonormal, we have $\sum_{j=1}^{3} \left(\vec{b}_j \frac{d_1^*\vec{W}}{dt}\right)\vec{b}_j = \frac{d_1^*\vec{W}}{dt}$, hence [4.19].

- Proof of [4.20]. Let us start from definition [4.2] for the virtual derivative of a vector with respect to a reference frame and let us use $[1.23]_a$ to write $\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{02}.\bar{\bar{Q}}_{21}, \bar{\bar{Q}}_{10} = \bar{\bar{Q}}_{12}.\bar{\bar{Q}}_{20}$:

$$\begin{aligned}
\frac{d_1^*\vec{W}}{dt} &\equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{W}\right)\dot{q}_i^* \\
&= \bar{\bar{Q}}_{02}\bar{\bar{Q}}_{21}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{12}\bar{\bar{Q}}_{20}.\vec{W}\right)\dot{q}_i^* \text{ (ordinary derivatives with respect to } q_i) \\
&= \bar{\bar{Q}}_{02}\bar{\bar{Q}}_{21}.\sum_{i=1}^{n}\left(\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}.\bar{\bar{Q}}_{20}.\vec{W} + \bar{\bar{Q}}_{12}.\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{20}.\vec{W})\right)\dot{q}_i^* \\
&= \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}_i^*\right).\bar{\bar{Q}}_{20}.\vec{W} + \bar{\bar{Q}}_{02}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{20}\vec{W})\dot{q}_i^* \\
&= \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}_i^*\right).\bar{\bar{Q}}_{20}.\vec{W} + \frac{d_2^*\vec{W}}{dt}
\end{aligned}$$

that is

$$\frac{d_1^*\vec{W}}{dt} = \frac{d_2^*\vec{W}}{dt} + \bar{\bar{\Omega}}_{12}^*.\vec{W} \qquad \text{by denoting } \bar{\bar{\Omega}}_{12}^* \equiv \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}_i^*\right).\bar{\bar{Q}}_{20}$$

What remains is to verify whether

$$\bar{\bar{\Omega}}_{12}^* \equiv \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}_i^*\right).\bar{\bar{Q}}_{20} = \bar{\bar{Q}}_{02}\bar{\bar{Q}}_{21}.\left(\sum_{i=1}^{n}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}_i^*\right).\bar{\bar{Q}}_{20}$$

is indeed skew-symmetric. By differentiating the identity $\bar{\bar{Q}}_{21}\bar{\bar{Q}}_{12} = \bar{\bar{I}}$ with respect to q_i, we find $\frac{\partial\bar{\bar{Q}}_{21}}{\partial q_i}\bar{\bar{Q}}_{12} + \bar{\bar{Q}}_{21}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i} = \bar{\bar{0}}$, hence $\frac{\partial\bar{\bar{Q}}_{21}}{\partial q_i}\bar{\bar{Q}}_{12} = -\bar{\bar{Q}}_{21}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i} = (\bar{\bar{Q}}_{21}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i})^T$, i.e. $\bar{\bar{Q}}_{21}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}$ is skew-symmetric. From this, it follows that the tensor $\bar{\bar{\Omega}}_{12}^*$ is skew-symmetric, by noting that if a tensor $\bar{\bar{A}}$ is skew-symmetric, then $\forall\bar{\bar{B}}$, the tensor $\bar{\bar{B}}^T\bar{\bar{A}}\bar{\bar{B}}$ is also skew-symmetric (here, $\bar{\bar{A}} = \bar{\bar{Q}}_{21}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}$ and $\bar{\bar{B}} = \bar{\bar{Q}}_{20}$). ■

To get an explicit expression for $\vec{\Omega}_{12}^*$ in [4.19], let us write $[\vec{x}_2, \vec{y}_2, \vec{z}_2]$ instead of $[\vec{b}_1, \vec{b}_2, \vec{b}_3]$. We then have

$$\boxed{\vec{\Omega}_{12}^* = \frac{1}{2}\left(\vec{x}_2 \times \frac{d_1^*\vec{x}_2}{dt} + \vec{y}_2 \times \frac{d_1^*\vec{y}_2}{dt} + \vec{z}_2 \times \frac{d_1^*\vec{z}_2}{dt}\right)}$$

Theorem. The virtual angular velocity vector $\vec{\Omega}^*_{12}$ of R_2 with respect to R_1 can be written as a linear form of the $\dot{q}^*_i$:

$$\boxed{\vec{\Omega}^*_{12} = \sum_{i=1}^{n} \vec{\omega}^i_{12}(q,t)\dot{q}^*_i} \qquad [4.21]$$

where the $\vec{\omega}^i_{12}$ are *the partial angular velocities of R_2 with respect to R_1* defined by [2.36] (these are the real velocities related to the real angular velocity $\vec{\Omega}_{12}$).

PROOF. According to definition [4.2], we have

$$\frac{d^*_1\vec{b}_j}{dt} \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right)\dot{q}^*_i$$

The proof is achieved by inserting this expression into definition [4.19] for $\vec{\Omega}^*_{12}$. ■

As was seen in section 2.7, some vectors $\vec{\omega}^i_{12}$ in [4.21] may be zero.

Analogy between real angular velocity and virtual angular velocity

The term *virtual angular velocity* in [4.19] is used to emphasize the analogy with the real angular velocity [1.48]. However, apart from this analogy, the virtual angular velocity has no connection with the real angular velocity, inasmuch as the $\dot{q}^*_i$ in [4.21] are arbitrary quantities.

From real angular velocity to virtual angular velocity [4.22]

Expressions [2.35] and [4.21] clearly demonstrate how to obtain the virtual angular velocity $\vec{\Omega}^*_{12}$ when the analytical expression for the real angular velocity $\vec{\Omega}_{12}$ is known:

- in the expression [2.35] for the real angular velocity, the term $\vec{\omega}^t_{12}$ (i.e. all terms that are not coefficients of a $\dot{q}_i$) is deleted
- and the $\dot{q}_i$ are replaced by arbitrary scalars, denoted by $\dot{q}^*_i$.

This procedure is analogous to the procedure used to obtain $\vec{V}^*_{12}$ from $\vec{V}_{12}$.

Moving from the real angular velocity tensor $\bar{\bar{\Omega}}_{12}$ [1.49] to the virtual angular velocity tensor $\bar{\bar{\Omega}}^*_{12}$ [4.20] is done in a similar manner:

$$\bar{\bar{\Omega}}_{12} = \bar{\bar{Q}}_{01}.\frac{d\bar{\bar{Q}}_{12}}{dt}.$$

$$\bar{\bar{Q}}_{20} = \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}_i + \frac{\partial\bar{\bar{Q}}_{12}}{\partial t}\right).\bar{\bar{Q}}_{20} \quad \rightsquigarrow \quad \bar{\bar{\Omega}}^*_{12} = \bar{\bar{Q}}_{01}.\left(\sum_{i=1}^{n}\frac{\partial\bar{\bar{Q}}_{12}}{\partial q_i}\dot{q}^*_i\right).\bar{\bar{Q}}_{20}$$

EXAMPLES.

1. Assume that the retained parameters in the problem are the Euler angles ψ, θ, φ defined in [2.3]. The angular velocity of the reference frame R_S defined by the rigid body S with respect to R_0 is

$$\vec{\Omega}_{R_0R_S} = \dot{\psi}\vec{z}_0 + \dot{\theta}\vec{n} + \dot{\varphi}\vec{z}_S$$

From this, we deduce that the virtual angular velocity of R_S with respect to R_0 is

$$\vec{\Omega}^*_{R_0R_S} = \dot{\psi}^*\vec{z}_0 + \dot{\theta}^*\vec{n} + \dot{\varphi}^*\vec{z}_S \qquad [4.23]$$

2. Now assume that the primitive parameters of the problem are the Euler angles ψ, θ, φ and that there exists a primitive constraint equation that is written as $\varphi = \omega t$, where ω is a given constant. The retained parameters are, therefore, ψ, θ, t, with the dependence on time t occurring via $\varphi = \omega t$.

   In this case, the angular velocity of the reference frame R_S defined by the rigid body S with respect to R_0 is

$$\vec{\Omega}_{R_0 R_S} = \dot{\psi}\vec{z}_0 + \dot{\theta}\vec{n} + \omega\vec{z}_S$$

   From this, it can be derived that the virtual angular velocity of R_S with respect to R_0 is

$$\vec{\Omega}^*_{R_0 R_S} = \dot{\psi}^*\vec{z}_0 + \dot{\theta}^*\vec{n}$$

3. Let us return to the example of the rotating bar S considered in section 2.5 and calculate the virtual angular velocities of the bar using two different parameterizations *ibid*.

   Recall that the primitive parameters of the bar are the same for the two parameterizations under consideration: the four coordinates X, Y, x, y and the two angles φ, θ.

   (a) First parameterization: Let us classify the constraint equation $\varphi = \omega t$ as a complementary equation, such that the retained parameters for the bar are $q = (X, Y, x, y, \varphi, \theta)$.

      i. The angular velocity of the reference frame R_S defined by the bar S with respect to R_1 is $\vec{\Omega}_{R_1 R_S} = \dot{\theta}\vec{z}_0$. There follows the virtual angular velocity of R_S with respect to R_1:

$$\vec{\Omega}^*_{R_1 R_S} = \dot{\theta}^*\vec{z}_0 \qquad [4.24]$$

      ii. For comparison's sake, let us calculate the virtual angular velocity with respect to R_0. From $\vec{\Omega}_{R_0 R_S} = (\dot{\varphi} + \dot{\theta})\vec{z}_0$, one finds that the virtual angular velocity of R_S in R_0 is

$$\vec{\Omega}^*_{R_0 R_S} = (\dot{\varphi}^* + \dot{\theta}^*)\vec{z}_0 \qquad [4.25]$$

   (b) Second parameterization: In this parameterization, the constraint equation $\varphi = \omega t$ is classified as a primitive and not as a complementary equation. The retained parameters are thus $q = (X, Y, x, y, \theta)$ and t, with the dependence on t occurring via $\varphi = \omega t$.

      i. The angular velocity of the reference frame R_S defined by the bar S with respect to R_1 is the same as in the first parameterization: $\vec{\Omega}_{R_1 R_S} = \dot{\theta}\vec{z}_0$. Therefore, the same goes for the virtual angular velocity:

$$\vec{\Omega}^*_{R_1 R_S} = \dot{\theta}^*\vec{z}_0 \qquad [4.26]$$

      ii. To calculate the virtual angular velocity with respect to R_0, let us start from the real angular velocity, which here is written as $\vec{\Omega}_{R_0 R_S} = (\omega + \dot{\theta})\vec{z}_0$. From this, it is found that the virtual angular velocity of R_S with respect to R_0 is

$$\vec{\Omega}^*_{R_0 R_S} = \dot{\theta}^*\vec{z}_0 \qquad [4.27]$$

      This expression is different from the expression that resulted from the first parameterization, which proves that the virtual angular velocity depends, *a priori*, on the chosen parameterization. ■

Dependence of the virtual angular velocity on time

According to [4.19], the virtual angular velocity $\vec{\Omega}^*_{12}$ contains the vectors

$$\frac{d^*_{R_1}\vec{b}_j}{dt} \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\vec{b}_j\right)\dot{q}^*_i \qquad [4.28]$$

where $\vec{b}_j$ and $\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{10}^{-1}$ depend, *a priori*, on time. Thus, $\vec{\Omega}^*_{12}$ depends *a priori* on t and should be denoted by $\vec{\Omega}^*_{12}(t)$. If we examine [4.20], it can be seen that the virtual angular velocity depends, *a priori*, on t via $\bar{\bar{Q}}_{01}(t), \bar{\bar{Q}}_{12}(t), \bar{\bar{Q}}_{20}(t)$ and we should once again write $\vec{\Omega}^*_{12}(t)$.

Nonetheless, similar to what happens to virtual velocities, it is seen that even if the virtual angular velocity does depend on t, this dependence has no effect on the theory. This is why we decide to *drop the argument* t *in* $\vec{\Omega}^*_{12}(t)$ and simply write $\vec{\Omega}^*_{12}$.

Dependence of the virtual angular velocity on the reference frame

Expressions [4.24]–[4.27] obtained in the above example clearly show that the virtual angular velocity $\vec{\Omega}^*_{R_1R_2}$ of R_2 with respect to R_1, defined in [4.19], depends, *a priori*, on the two reference frames R_1, R_2.

As concerns the dependence on R_1, it is possible to make the same observations as for the virtual velocities, namely that one should distinguish between the expression for the virtual angular velocity $\vec{\Omega}^*_{R_1R_2}$ and its value:

1. *The expression* [4.19] for $\vec{\Omega}^*_{12}$ does indeed depend on the reference frame R_1 (for example, compare [4.24] and [4.25]).

2. Things are different for the *value* of $\vec{\Omega}^*_{12}$. When we give the $\dot{q}^*_i$ all the values in the set of real numbers $\mathbb{R}$ (recall that the $\dot{q}^*_i$ in [4.21] or [4.28] are arbitrary), it may be that the set of values taken by the vector $\vec{\Omega}^*_{R_1R_2}$ is independent of R_1. For example:

   – when $\dot{\theta}^*$ in [4.24] takes all the values in $\mathbb{R}$, the vector $\vec{\Omega}^*_{R_1R_2}$ spans the entire vector axis $\mathbb{R}\vec{z}_0$,

   – when $\dot{\theta}^*, \dot{\varphi}^*$ in [4.25] take all the values in $\mathbb{R}$, the vector $\vec{\Omega}^*_{R_1R_2}$ spans the same vector axis.

   To express this fact, we say that *the value* of $\vec{\Omega}^*_{R_1R_2}$ may be independent of the reference frame R_1.

It is important to retain the index R_1 in the notation $\vec{\Omega}^*_{R_1R_2}$ for the virtual velocity. As we will see later on, this enables one to write the composition formulae of the virtual angular velocities analogous to those for the real angular velocities, for example [4.45]: $\vec{\Omega}^*_{13} = \vec{\Omega}^*_{12} + \vec{\Omega}^*_{23}$.

Here are the cases where the expression for the virtual angular velocity does not depend on the reference frame with respect to which the virtual angular velocity is calculated:

Theorem.

1. HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

   Then, the virtual angular velocity $\vec{\Omega}^*_{12}$ is written as

$$\boxed{\vec{\Omega}^*_{12} = \frac{1}{2}\sum_{j=1}^{3}\vec{b}_j \times \left(\sum_{i=1}^{n}\frac{\partial\vec{b}_j}{\partial q_i}\dot{q}^*_i\right)} \qquad [4.29]$$

The virtual angular velocity $\vec{\Omega}^*_{12}$ *no longer depends on* R_1, in the sense that if R_1 is replaced by any other reference frame, we still arrive *at the same expression* for the virtual angular velocity. It depends only on R_2.

2. HYPOTHESIS: The rotation tensor $\bar{\bar{Q}}_{12}$ does *not* depend on q.

   Then

$$\boxed{\vec{\Omega}^*_{12} = \vec{0}} \qquad [4.30]$$

FIRST PROOF OF [4.30].

1. Let us assume that $\bar{\bar{Q}}_{01}$ does not depend on q. Relationship [4.28] becomes

$$\frac{d^*_{R_1}\vec{b}_j}{dt} = \underbrace{\bar{\bar{Q}}_{01}.\bar{\bar{Q}}_{10}}_{=\bar{\bar{I}}}.\sum_{i=1}^{n}\frac{\partial\vec{b}_j}{\partial q_i}\dot{q}^*_i,$$

   which is the same relationship that would be obtained by applying [4.9]. By inserting this relationship into [4.19], we obtain [4.29].

2. If $\bar{\bar{Q}}_{12}$ does not depend on q, relationship [4.20] immediately gives $\bar{\bar{\Omega}}^*_{12} = \bar{\bar{0}}$, thus $\vec{\Omega}^*_{12} = \vec{0}$.

SECOND PROOF OF [4.30] (a longer proof). We can also prove $\vec{\Omega}^*_{12} = \vec{0}$ by starting from the above-obtained [4.29], provided that we also adopt hypothesis [2.26]. As the rotation tensors $\bar{\bar{Q}}_{01}$ and $\bar{\bar{Q}}_{12}$ do not depend on q, it is the same for the tensor $\bar{\bar{Q}}_{02} = \bar{\bar{Q}}_{01}.\bar{\bar{Q}}_{12}$.

As $\bar{\bar{Q}}_{02}$ does not depend on q, relationship [4.29] can be transformed as follows:

$$\vec{\Omega}^*_{12} = \frac{1}{2}\sum_{j=1}^{3}\vec{b}_j \times \left(\bar{\bar{Q}}_{02}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}(\bar{\bar{Q}}_{20}.\vec{b}_j)\dot{q}^*_i\right)$$

As the vectors $\vec{b}_j$ are fixed in R_2, the vectors $\bar{\bar{Q}}_{20}\vec{b}_j$ are constant vectors of E (see definition [1.35]). Consequently, $\dfrac{\partial}{\partial q_i}(\bar{\bar{Q}}_{20}.\vec{b}_j) = \vec{0}$. ■

EXAMPLE. In the preceding example, if we use the second parameterization, hypothesis [4.29] is verified. Relationships [4.26] and [4.27] show that the expression for the virtual angular velocity is the same in both reference frames R_1 and R_0. ■

The virtual derivation with respect to R_1 of a vector constant in R_2

The following theorem is an important particular case of [4.19]:

Theorem. The virtual derivation formula with respect to R_1 of a vector constant in R_2.

$$\forall R_1, R_2, \forall\vec{W} \in E \text{ constant in } R_2, \boxed{\frac{d^*_1\vec{W}}{dt} = \vec{\Omega}^*_{12} \times \vec{W}} \qquad [4.31]$$

PROOF. We have just to apply [4.19] observing that, since $\vec{W}$ is constant in R_2, we have $\dfrac{d^*_2\vec{W}}{dt} = \vec{0}$ according to [4.4]. ■

4.4. Virtual velocities in a rigid body

The results given in this section are valid for *a rigid body*.

Theorem. $\forall t$, $\forall R_1$, $\forall$ *rigid body* S defining a reference frame R_S, $\forall$ particles p, p' belonging to the rigid body S, whose respective positions in R_0 are P, P', the VV [4.10] verifies the following relationship

$$\boxed{\vec{V}^*_{R_1}(p') = \vec{V}^*_{R_1}(p) + \vec{\Omega}^*_{R_1 R_S} \times \overrightarrow{PP'}} \qquad [4.32]$$

where the virtual angular velocity vector $\vec{\Omega}^*_{R_1 R_S}$ of R_S with respect to R_1 is given by [4.19] or [4.21].

PROOF. The reasoning is similar to that used in [1.61] for real velocities. Let O_1 be a point of $\mathcal{E}$ fixed in R_1. We have

$$\begin{aligned}\vec{V}^*_{R_1}(p') - \vec{V}^*_{R_1}(p) &= \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\Big(\bar{\bar{Q}}_{10}.\underbrace{(\overrightarrow{O_1P'} - \overrightarrow{O_1P})}_{\overrightarrow{PP'}}\Big)\dot{q}_i^* && \text{according to definition [4.10]}\\ &= \frac{d^*_{R_1}\overrightarrow{PP'}}{dt} && \text{according to definition [4.2]}\end{aligned}$$

As the particles p, p' belong to the rigid body S, their respective positions in R_S, $P^{(S)}$ and $P'^{(S)}$, are fixed points in $\mathcal{E}$ over time. Thus, $\forall t$, $\overrightarrow{P^{(S)}P'^{(S)}}$ is a constant vector of E. Further, according to [1.22], we have $\overrightarrow{P^{(S)}P'^{(S)}} = \bar{\bar{Q}}_{S0}.\overrightarrow{PP'}$ and, thus, $\bar{\bar{Q}}_{S0}.\overrightarrow{PP'}$ is a constant vector in E. In other words, the vector $\overrightarrow{PP'}$ is constant in R_S according to definition [1.35].

We thus have $\dfrac{d^*_{R_1}\overrightarrow{PP'}}{dt} = \vec{\Omega}^*_{R_1 R_S} \times \overrightarrow{PP'}$ by applying [4.31]. ■

4.4.1. *The virtual velocity field (VVF) associated with a parameterization*

The notation $\vec{V}^*(p)$ is natural and easy to understand. The only problem it poses is that it contains, as an argument, a particle. Because of this, we cannot speak of velocity *fields*, whose arguments are *points* in the affine space $\mathcal{E}$. The following new notation, which is said to be Eulerian, is a little more complex, but enables us to overcome this problem.

Eulerian (or spatial) notation. Consider a rigid body S and a point $A \in pos_{R_0}(S,t)$. We write

$$\boxed{\vec{V}^*_{R_1 S}(A) \equiv \begin{array}{l}\text{the virtual velocity with respect to } R_1\\ \text{of the particle of } S \text{ passing through point } A \text{ at instant } t\end{array}} \qquad [4.33]$$

When using the Eulerian notation, the particle is not known by its name, but by its position at the instant considered. In general, the particle is *not* the same over the course of time.

Let p be a particle of S whose position in R_0, over the course of time, is $P = pos_{R_0}(p,t)$. We have the trivial equality

$$\forall t, \qquad \vec{V}^*_{R_1 S}(P) = \vec{V}^*_{R_1}(p) \underset{[4.10]}{\equiv} \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)\dot{q}_i^*$$

On the contrary, for any point $A \in pos_{R_0}(S,t)$:

$$\vec{V}^*_{R_1 S}(A) \neq \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}\right)\dot{q}_i^*$$

The equality only holds when A is a point *attached to the rigid body* $\mathcal{S}$, i.e., when it designates the position of a *same* particle in the rigid body over time.

• With the help of the Eulerian notation [4.33], we can define the following:

Definition. *The VVF of a rigid body S (with respect to R_1 and at instant t), associated with* (or *resulting from*) *the parameterization* [2.19], is defined as

$$\begin{array}{rcl} V^*_{R_1 S} : \ pos_{R_0}(S,t) & \to & E \\ A & \mapsto & \vec{V}^*_{R_1 S}(A) \end{array} \qquad [4.34]$$

The VVF is associated with parameterization [2.19], but this will not be repeated systematically, for brevity.

REMARK. Similar to the remark after definition [4.10], definition [4.34] contains $(\dot{q}^*_1, \ldots, \dot{q}^*_n)$, which is an arbitrary n-tuple in $\mathbb{R}^n$.

– To emphasize this arbitrary character, we can say that [4.34] defines (the expression for) *the most general VVF* of the rigid body S, or, more concisely *the VVF*.

– If we work with a given n-tuple $(\dot{q}^*_1, \ldots, \dot{q}^*_n)$, we speak of *a VVF*.

If we are looking at values, we can speak of *virtual velocity fields* or of *a virtual velocity field*:

– when $(\dot{q}^*_1, \ldots, \dot{q}^*_n)$ is given all the values in $\mathbb{R}^n$, this generates *the set of VVFs*,

– each value taken by the n-tuple $(\dot{q}^*_1, \ldots, \dot{q}^*_n)$ corresponds to *one VVF* of the rigid body S.

In this book, we will have to work with one of the above meanings. However, we will not distinguish between them at all times. We will use "the VVF" or "a VFF" interchangeably, knowing that the precise meaning will be provided by the context. ■

4.4.2. *Virtual velocity field (VVF) in a rigid body*

Theorem. $\forall t, \forall R_1, \forall$ *rigid body* S defining a reference frame R_S, $\forall A, B \in pos_{R_0}(S,t) \subset \mathcal{E}$,

$$\boxed{\vec{V}^*_{R_1 S}(B) = \vec{V}^*_{R_1 S}(A) + \vec{\Omega}^*_{R_1 R_S} \times \overrightarrow{AB}} \qquad [4.35]$$

Thus, $\forall t$, the VVF $V^*_{R_1 S}$ (defined over $pos_{R_0}(S,t)$) is completely determined by the virtual velocity at one point (here, point A) and the virtual angular velocity $\vec{\Omega}^*_{R_1 R_S}$. (If the rigid body S is rectilinear, the virtual angular velocity $\vec{\Omega}^*_{R_1 R_S}$ is determined to within an arbitrary vector collinear to S.)

PROOF. The proof is similar to that of [1.63] for the real velocities. Let t be a fixed instant, A, B two given points $\in pos_{R_0}(S,t) \subset \mathcal{E}$. *At the instant* t,

• the point A is the position $P(q,t)$ of a particle p of the rigid body S with respect to R_0,

• the point B is the position $P'(q,t)$ of a particle p' of the rigid body S with respect to R_0.

We thus have *at the instant* t

$$\vec{V}^*_{R_1 S}(A) = \vec{V}^*_{R_1}(p) \qquad \vec{V}^*_{R_1 S}(B) = \vec{V}^*_{R_1}(p') \qquad [4.36]$$

The proof is achieved by applying [4.32] at instant t and with the particles p, p' of S that were just defined. ■

The equalities [4.36] hold only at the instant t, but this is sufficient for the proof.

Let us reiterate a comment that was already made with respect to the real velocity field. For mathematical convenience, we may define the virtual velocity field over the entire space $\mathcal{E}$ and not only over $pos_{R_0}(S,t)$. To do this, we carry out a classical operation in rigid body mechanics, which consists of "extending" the rigid body S "to infinity", so as to be able to state the previous theorem with the VVF of a rigid body S defined over all of $\mathcal{E}$. However, this was not done.

4.5. Virtual velocities in a system

4.5.1. *VVF associated with a parameterization*

The Eulerian notation [4.33] can easily be generalized to a system:

Eulerian notation. Consider a system $\mathcal{S}$ and a point $A \in pos_{R_0}(\mathcal{S},t)$. We write

$$\boxed{\vec{V}^*_{R_1\mathcal{S}}(A) \equiv \begin{array}{l}\text{the virtual velocity with respect to } R_1\\ \text{of the particle of } \mathcal{S} \text{ passing through point } A \text{ at instant } t\end{array}} \qquad [4.37]$$

When using Eulerian notation, the particle is not known by its name, but by its position at the instant considered. In general, the particle is *not* the same over the course of time.

As with real velocities, note that there exists a case when the notation $\vec{V}^*_{R_1\mathcal{S}}(A)$ is ambiguous. This is when A is the contact point I between two rigid bodies S_i and S_j in $\mathcal{S}$ (refer again to Figure 1.9). In this case, we must distinguish between the two virtual velocities $\vec{V}^*_{R_1S_i}(I)$ and $\vec{V}^*_{R_1S_j}(I)$ of the two particles of S_i and S_j respectively, passing through the same point I at the instant t considered. These particles are infinitely close but are not identical.

In general, the point I is attached to neither S_i nor S_j and

$$\vec{V}^*_{R_1S_i}(I) \neq \vec{V}^*_{R_1S_j}(I)$$

We can see the importance of the second index, specifying the rigid body, in the notation for the virtual velocity vector.

• Let p be a particle of $\mathcal{S}$ of position $P = pos_{R_0}(p,t)$ in R_0 over time. We have the trivial equality

$$\forall t, \qquad \vec{V}^*_{R_1\mathcal{S}}(P) = \vec{V}^*_{R_1}(p) \underset{[4.10]}{\equiv} \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)\dot{q}_i^*$$

On the contrary, for any point $A \in pos_{R_0}(\mathcal{S},t)$ that is not attached to the system $\mathcal{S}$:

$$\vec{V}^*_{R_1\mathcal{S}}(A) \neq \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}\right)\dot{q}_i^*$$

The equality holds only when A is a point that is *attached to the system* $\mathcal{S}$, i.e., when A denotes the position of a *same* particle of the system over time.

• By means of the Eulerian notation [4.37], it is possible to define the following:

Definition. *The VVF of a system $\mathcal{S}$ (with respect to R_1 and at an instant t), associated with* (or *resulting from*) *the parameterization* [2.19], is defined as

$$\begin{array}{rccc} V^*_{R_1\mathcal{S}}: & pos_{R_0}(\mathcal{S},t) & \rightarrow & E \\ & A & \mapsto & \vec{V}^*_{R_1\mathcal{S}}(A) \end{array} \qquad [4.38]$$

The VVF is associated with parameterization [2.19], but this will not be repeated throughout the text, for brevity.

It should be noted that for a system $\mathcal{S}$ made up of several rigid bodies, the VVF $V^*_{R_1\mathcal{S}}$ does *not* satisfy a relation of the type [4.35].

4.5.2. VVF on each rigid body of a system

The following result can be readily derived from [4.35]:

Theorem. The VVF [4.38], *restricted to each rigid body S* of a system $\mathcal{S}$, satisfies relation [4.35] at any instant t. [4.39]

4.5.3. Virtual velocity of the center of mass

We present below a result that is specific to the center of mass of a system:

Theorem and definition. Let $\mathcal{S}$ be a system of mass m and of mass center G. We have

$$\forall R_1, \forall \text{ VVF } V^*_{R_1}, \qquad \boxed{\int_{\mathcal{S}} \vec{V}^*_{R_1}(p)dm = m\vec{V}^*_{R_1\mathcal{S}}(G)} \qquad [4.40]$$

where $\vec{V}^*_{R_1\mathcal{S}}(G)$ is defined by (O_1 being a fixed point in R_1)

$$\boxed{\vec{V}^*_{R_1\mathcal{S}}(G) = \frac{d^*_{R_1}\overrightarrow{O_1G}}{dt} \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1G}\right)\dot{q}^*_i} \qquad [4.41]$$

• If we adopt the following hypothesis:

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

Then, expression [4.41] for $\vec{V}^*_{R_1\mathcal{S}}(G)$ becomes

$$\boxed{\vec{V}^*_{R_1\mathcal{S}}(G) \equiv \sum_{i=1}^{n}\frac{\partial\overrightarrow{O_1G}}{\partial q_i}\dot{q}^*_i} \qquad [4.42]$$

• As a matter of fact, if $\mathcal{S}$ is a rigid body, then the mass center G is a point attached to the rigid body and expressions [4.41] and [4.42] for $\vec{V}^*_{R_1\mathcal{S}}(G)$ are not new, they are a consequence of [4.10] and [4.11], respectively. On the contrary, if $\mathcal{S}$ is a system composed of several rigid bodies, then the center G is not attached to $\mathcal{S}$ and expressions [4.41] and [4.42] must be understood as specific definitions designed for G.

FIRST PROOF. According to [4.10], we have

$$\int_{\mathcal{S}} \vec{V}^*_{R_1}(p)dm = \int_{\mathcal{S}} \bar{\bar{Q}}_{01}(q,t).\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}(q,t).\overrightarrow{O_1P}\right)\dot{q}^*_i dm$$

On the right-hand side, the integration variable P sweeps over the position of the system $\mathcal{S}$ defined by (q,t) and it is *independent* of (q,t). The integral sign can, thus, be entered into the sum $\sum_{i=1}^{n}$ and then under the derivative $\frac{\partial}{\partial q_i}$ as follows

$$\int_{\mathcal{S}} \vec{V}^*_{R_1}(p)dm = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\int_{\mathcal{S}} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right) dm\, \dot{q}_i^* = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\Big(\bar{\bar{Q}}_{10}.\underbrace{\int_{\mathcal{S}}\overrightarrow{O_1P}dm}_{m\overrightarrow{O_1G}}\Big)\dot{q}_i^*$$

Hence [4.40] and [4.41]. Finally, expression [4.42] follows immediately from [4.41] taking account of the hypothesis in the theorem statement. ■

SECOND PROOF. The previous proof is a little long because we are working with the general expression [4.10] for the VV $\vec{V}^*_{R_1}(p)$. The proof is simplified if, from the start, we use hypothesis [2.26]. Indeed, the VV $\vec{V}^*_{R_1}(p)$ is then given by [4.11] and we have

$$\int_{\mathcal{S}} \vec{V}^*_{R_1}(p)dm = \int_{\mathcal{S}}\sum_{i=1}^{n}\frac{\partial\overrightarrow{O_1P}}{\partial q_i}\,\dot{q}_i^* dm$$

The same argument as in the previous proof allows one to put the integral sign behind the sum and then under the derivative with respect to q_i:

$$\int_{\mathcal{S}} \vec{V}^*_{R_1}(p)dm = \sum_{i=1}^{n}\frac{\partial}{\partial q_i}\underbrace{\int_{\mathcal{S}}\overrightarrow{O_1P}dm}_{m\overrightarrow{O_1G}}\,\dot{q}_i^* = m\sum_{i=1}^{n}\frac{\partial\overrightarrow{O_1G}}{\partial q_i}\dot{q}_i^*\text{: this is [4.42].}\blacksquare$$

4.6. Composition of virtual velocities

4.6.1. *Composition of virtual velocities of a particle*

Theorem and definition. $\forall R_1, R_2, \forall$ particle $p, \forall t$:

$$\boxed{\vec{V}^*_{R_1}(p) = \vec{V}^*_{R_1S(R_2)}(P) + \vec{V}^*_{R_2}(p)} \quad \text{or in shortened form:} \quad \boxed{\vec{V}^*_1(p) = \vec{V}^*_{12}(P) + \vec{V}^*_2(p)} \qquad [4.43]$$

where we recall that $\vec{V}^*_{R_1S(R_2)}(P) \equiv \vec{V}^*_{12}(P)$ is the virtual velocity with respect to R_1 of the particle of $S(R_2)$ (that is, the particle attached to R_2), which coincides with $P(t)$ at the instant t. As in [1.70], this velocity is called *the background virtual velocity*.

PROOF. Let us choose a point $O_1 \in \mathcal{E}$ fixed in R_1 and a point $O_2 \in \mathcal{E}$ fixed in R_2. We have

$$\vec{V}^*_{R_1}(p) = \frac{d_1^*\overrightarrow{O_1P}}{dt} = \frac{d_1^*\overrightarrow{O_1O_2}}{dt} + \frac{d_1^*\overrightarrow{O_2P}}{dt}$$

On the one hand, $\frac{d_1^*\overrightarrow{O_1O_2}}{dt} = \vec{V}^*_{R_1}(o_2)$, where o_2 denotes the particle located at O_2 at any instant. On the other hand, relationship [4.19] gives $\frac{d_1^*\overrightarrow{O_2P}}{dt} = \frac{d_2^*\overrightarrow{O_2P}}{dt} + \vec{\Omega}^*_{12}\times\overrightarrow{O_2P} = \vec{V}^*_{R_2}(p) + \vec{\Omega}^*_{12}\times\overrightarrow{O_2P}$. Hence

$$\vec{V}^*_{R_1}(p) = \vec{V}^*_{R_1}(o_2) + \vec{\Omega}^*_{12}\times\overrightarrow{O_2P} + \vec{V}^*_{R_2}(p)$$

Let us introduce the particle p_2 of $S(R_2)$, whose movement in R_0 is $\tau \mapsto P_2(\tau)$, such that at the instant t $P_2(t) = P(t)$; in other words, such that the particle p_2 passes through P at t. According to [4.32], we have: $\vec{V}^*_{R_1}(o_2) + \vec{\Omega}^*_{12} \times \overrightarrow{O_2P} = \vec{V}^*_{R_1}(p_2)$. ■

Theorem.

HYPOTHESIS [2.33] adopted both for R_1 and R_2: The rotation tensors $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and $\bar{\bar{Q}}_{02}$ of R_2 with respect to R_0, as well as the points O_1 and O_2, fixed in R_1 and R_2 respectively, do *not* depend on q.

Then, [4.43] becomes

$$\boxed{\vec{V}^*_{R_1}(p) = \vec{V}^*_{R_2}(p) \underset{[4.18]}{=} \sum_{i=1}^{n} \frac{\partial \overrightarrow{P}}{\partial q_i} \dot{q}_i^*} \quad \Leftrightarrow \quad \boxed{\vec{V}^*_{R_1 S(R_2)}(P) = \vec{0}} \qquad [4.44]$$

PROOF. The hypothesis adopted allows one to apply theorem [4.18]: the virtual velocity $\vec{V}^*_{R_1}(p)$ (respectively $\vec{V}^*_{R_2}(p)$) does not depend on R_1 (respectively R_2). Hence, $\vec{V}^*_{R_1}(p) = \vec{V}^*_{R_2}(p)$. It results from [4.43] that $\vec{V}^*_{R_1 S(R_2)}(P) = \vec{0}$. ■

SECOND PROOF. We can directly prove the second equality in [4.44]; namely, $\vec{V}^*_{R_1 S(R_2)}(P) = \vec{0}$, without using [4.43]. To do this, let a be the particle of $S(R_2)$ passing through P at the instant t considered, and let A be its position (in R_0) at any instant. The point A is located at the point P at the instant t. We have

$$\begin{aligned} \vec{V}^*_{R_1 S(R_2)}(P) &= \vec{V}^*_{R_2}(a) && \text{(equality valid only at the considered instant } t) \\ &= \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1A}}{\partial q_i} \dot{q}_i^* && \text{according to [4.11], which holds as } \bar{\bar{Q}}_{01} \text{ is assumed to be} \\ & && \text{independent of } q. \end{aligned}$$

Let us write $\overrightarrow{O_1A} = \overrightarrow{O_1O_2} + \overrightarrow{O_2A}$. According to the hypothesis in the theorem, $\overrightarrow{O_1O_2}$ and $\overrightarrow{O_2A}$ do not depend on q and, therefore, neither does $\overrightarrow{O_1A}$. Consequently, $\forall i$, $\dfrac{\partial \overrightarrow{O_1A}}{\partial q_i} = \vec{0}$ and thus $\vec{V}^*_{R_1 S(R_2)}(P) = \vec{0}$. Note that all this reasoning is carried out at a given instant t. ■

In Chapter 5, it will be seen that the virtual power [5.14] of the inter efforts between two rigid bodies S_1 and S_2 involves the virtual velocity field $V^*_{R_2 S_1}$, where R_2 is the reference frame defined by the rigid body S_2. The hypothesis in the previous theorem is *not* verified for this virtual power; the VVF $V^*_{R_2 S_1}$ is *not* zero and it plays an essential role in the VP of the inter efforts. Moreover, it is clear that in this kind of situation, the reference frame index, R_2, is absolutely necessary in the notation $V^*_{R_2 S_1}$.

4.6.2. *Composition of virtual angular velocities*

Theorem. Composition of the virtual angular velocities.

$$\forall R_1, R_2, R_3, \quad \boxed{\vec{\Omega}^*_{13} = \vec{\Omega}^*_{12} + \vec{\Omega}^*_{23}} \qquad [4.45]$$

PROOF. Let us apply [4.19] to the three pairs of reference frames (R_1, R_2), (R_2, R_3) and (R_1, R_3):

$$\forall \vec{W} \in E, \frac{d_1^* \vec{W}}{dt} = \frac{d_2^* \vec{W}}{dt} + \vec{\Omega}_{12}^* \times \vec{W} \qquad \frac{d_2^* \vec{W}}{dt} = \frac{d_3^* \vec{W}}{dt} + \vec{\Omega}_{23}^* \times \vec{W} \qquad \frac{d_1^* \vec{W}}{dt} = \frac{d_3^* \vec{W}}{dt} + \vec{\Omega}_{13}^* \times \vec{W} \qquad [4.46]$$

Taking the sum of the first two equalities yields

$$\frac{d_1^* \vec{W}}{dt} = \frac{d_3^* \vec{W}}{dt} + (\vec{\Omega}_{12}^* + \vec{\Omega}_{23}^*) \times \vec{W} \qquad [4.47]$$

By identifying [4.47] with the last equality in [4.46], we get $\vec{\Omega}_{13}^* \times \vec{W} = (\vec{\Omega}_{12}^* + \vec{\Omega}_{23}^*) \times \vec{W}$, hence [4.45], taking into account the fact that vector $\vec{W}$ is arbitrary. ■

• If the rotation tensor $\bar{\bar{Q}}_{12}$ does *not* depend on q, then $\vec{\Omega}_{12}^* = \vec{0}$ according to [4.30] and relationship [4.45] becomes $\vec{\Omega}_{13}^* = \vec{\Omega}_{23}^*$.

• By making $R_3 = R_1$ in [4.45], we immediately get the following:

Corollary. The virtual angular velocity vector of R_2 with respect to R_1 is the negative of the virtual angular velocity vector of R_1 with respect to R_2:

$$\boxed{\vec{\Omega}_{12}^* = -\vec{\Omega}_{21}^*} \qquad [4.48]$$

4.6.3. *Composition of VVFs in rigid bodies*

The composition of virtual velocities in a rigid body is obtained using the results that we just established for a particle and by applying this to each particle of the rigid body.

Theorem and definition. $\forall R_1, R_2, \forall$ rigid body S,

$$\boxed{V_{R_1 S}^* = V_{R_1 S(R_2)}^* + V_{R_2 S}^*} \quad \text{or in shortened form:} \quad \boxed{V_{1S}^* = V_{12}^* + V_{2S}^*} \qquad [4.49]$$

$V_{R_1 S(R_2)}^*$ is called *the VVF of R_2 relative to (or, with respect to) R_1*, or *the background VVF*. An immediate consequence of [4.49] is

$$\boxed{\vec{\Omega}_{1S}^* = \vec{\Omega}_{12}^* + \vec{\Omega}_{2S}^*} \qquad [4.50]$$

which is merely [4.45].

FIRST PROOF. Let us apply theorem [4.43] to a particle p of S, noting that if p is a particle of S, then $\forall R_i, \forall t, \vec{V}_{R_i}^*(p) = \vec{V}_{R_i S}^*(P)$:

$$\forall R_1, R_2, S, \forall \text{ particle } p \text{ of } S, \forall t : \vec{V}_{R_1 S}^*(P) = \vec{V}_{R_1 S(R_2)}^*(P) + \vec{V}_{R_2 S}^*(P)$$

At the instant t considered, this relationship is true $\forall$ particle p of S. In other words, it is true $\forall P(t)$, that is $\forall A$, hence $V_{1S}^* = V_{12}^* + V_{2S}^*$. ■

NOTE. The virtual field $V_{R_1 S(R_2)}^*$ can be derived from the real field $V_{R_1 S(R_2)}$ using the procedure described in section 4.2. The name given to $V_{R_1 S(R_2)}^*$ is, therefore, consistent. ■

Theorem. $\forall t, \forall R_1, R_2$, the virtual velocity field $V^*_{R_1S(R_2)}$ and $V^*_{R_2S(R_1)}$ are opposite:

$$\boxed{V^*_{R_1S(R_2)} = -V^*_{R_2S(R_1)}} \quad \text{or, in shortened form} \quad \boxed{V^*_{12} = -V^*_{21}} \qquad [4.51]$$

(Through [4.48] it is already known that the principal vectors of these fields, $\vec{\Omega}^*_{12}$ and $\vec{\Omega}^*_{21}$, are opposite.)

PROOF. All we need to do is apply [4.49] taking S equal to the rigid body $S(R_1)$ defined by R_1. ■

SECOND PROOF. Starting from [4.43]. This relation gives: $\forall$ particle p, $\forall t$:

$$\begin{cases} \vec{V}^*_{R_1}(p) &= \vec{V}^*_{R_1S(R_2)}(P) + \vec{V}^*_{R_2}(p) \\ \vec{V}^*_{R_2}(p) &= \vec{V}^*_{R_2S(R_1)}(P) + \vec{V}^*_{R_1}(p) \end{cases}$$

Adding these equalities gives

$$\vec{0} = \vec{V}^*_{R_1S(R_2)}(P) + \vec{V}^*_{R_2S(R_1)}(P)$$

At the instant t considered, this equality is true for any particle p, that is, for any point $P(t)$, hence $V^*_{12} = -V^*_{21}$. ■

By iterating the theorem [4.49], we arrive at the following:

Corollary 1. For any integer $m \geq 2$, we have the following relationship that generalizes [4.49]:

$$\forall R_1, R_2, \cdots, R_m, \forall \text{ rigid body } S, \boxed{V^*_{1S} = V^*_{12} + V^*_{23} + \cdots + V^*_{mS}} \qquad [4.52]$$

This entails the following relationship, which generalizes [4.50]:

$$\boxed{\vec{\Omega}^*_{1S} = \vec{\Omega}^*_{12} + \vec{\Omega}^*_{23} + \cdots + \vec{\Omega}^*_{mS}} \qquad [4.53]$$

Corollary 2. If the hypothesis in [4.44] is satisfied, then

$$\boxed{V^*_{R_1S} = V^*_{R_2S}} \qquad [4.54]$$

A consequence of this is

$$\boxed{\vec{\Omega}^*_{1S} = \vec{\Omega}^*_{2S}} \quad \text{i.e.} \quad \boxed{\vec{\Omega}^*_{12} = \vec{0}} \quad \text{(which is consistent with [4.30])}$$

PROOF. Just apply relationship [4.44] to [4.49]. ■

4.7. Method of calculating the virtual velocity at a point

Let $\mathcal{S}$ be a system composed of a finite numbers of rigid bodies and let A be a point in $pos_{R_0}(\mathcal{S}, t)$. To calculate the virtual velocity $\vec{V}^*_{R_1\mathcal{S}}(A)$, we distinguish between two cases:

1. If A is a point *attached to* $\mathcal{S}$, that is, if it is the position of the *same* particle a of the system over time, then

$$\forall t, \qquad \vec{V}^*_{R_1\mathcal{S}}(A) = \vec{V}^*_{R_1}(a) = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}\right)\dot{q}^*_i$$

2. If A is *not* attached to $\mathcal{S}$, then $\vec{V}^*_{R_1\mathcal{S}}(A) \neq \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1A}\right)\dot{q}_i^*$! We must calculate $\vec{V}^*_{R_1\mathcal{S}}(A)$ with the help of relationship [4.35], going by another point B whose virtual velocity is known. At a given instant t, let S_j be the rigid body of $\mathcal{S}$ such that $A \in pos_{R_0}(S_j, t)$, we then apply the quoted relationship to S_j:

$$\vec{V}^*_{R_1\mathcal{S}}(A) = \vec{V}^*_{R_1S_j}(A) = \vec{V}^*_{R_1S_j}(B) + \vec{\Omega}^*_{R_1S_j} \times \overrightarrow{BA}$$

The center of mass G of system $\mathcal{S}$ is an exception. Whether or not G is attached to $\mathcal{S}$, it is always possible to calculate $\vec{V}^*_{R_1\mathcal{S}}(G)$ using relationship [4.41]:

$$\vec{V}^*_{R_1\mathcal{S}}(G) \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1G}\right)\dot{q}_i^*$$

It has been seen that in the case when A is the contact point I between two rigid bodies S_i and S_j of $\mathcal{S}$ (Figure 1.9), we must distinguish between the two virtual velocities $\vec{V}^*_{R_1S_i}(I)$ and $\vec{V}^*_{R_1S_j}(I)$ of the two particles S_i and S_j respectively, passing through the same point I at the instant t considered.

In general, the point I is not attached to either S_i or S_j and we calculate each of the two virtual velocities $\vec{V}^*_{R_1S_i}(I)$ and $\vec{V}^*_{R_1S_j}(I)$ according to method no. 2 above. In general, we find

$$\vec{V}^*_{R_1S_i}(I) \neq \vec{V}^*_{R_1S_j}(I)$$

5

Virtual Powers

In this chapter, we will study the virtual powers (VP) of efforts and quantities of acceleration, which are *grosso modo* the product of efforts and the virtual velocities described in Chapter 4. These VP are the ingredients of the principle of virtual powers (PVP), which are an essential tool in analytical mechanics. We will also study the concept of "potential", the counterpart of which in Newtonian mechanics is potential energy.

5.1. Principle of virtual powers

PVP is a very general principle in mechanics, valid for any mechanical system. It is equivalent to Newton's laws of motion and yet it makes it easier to establish the equations of motion of a mechanical system. It is commonly used in analytical mechanics.

The PVP brings into play scalar functions that are called virtual powers (VP), which will be defined and studied in detail in this chapter. The PVP can be stated as follows:

Principle of virtual powers. $\exists$ a reference frame R_g called a *Galilean* (or *inertial*) reference frame such that $\forall$ system $\mathcal{S}$, $\forall t$, $\forall$ VVF with respect to R_g, the virtual power, with respect to R_g and at instant t, of the external and internal efforts in this VVF is equal to the VP, at instant t, of the quantities of acceleration with respect to R_g and in the same VVF:

$$\boxed{\mathscr{P}^*_{R_g}(\mathcal{F}_{ext\to\mathcal{S}}, t) + \mathscr{P}^*_{R_g}(\mathcal{F}_{int\to\mathcal{S}}, t) = \mathscr{P}^*_{R_g}(\rho\vec{\Gamma}_{R_g\mathcal{S}}, t)} \qquad [5.1]$$

The PVP holds for any mechanical system and any VVF. In the analytical mechanics framework, it will be applied to

- systems composed of rigid bodies, some of which may be reduced to particles,
- and the virtual velocity fields associated with parameterization [2.19]. As seen in [4.39], these VVF, defined by [4.34] or [4.38] and calculated using the virtual velocities [4.10] or [4.11], satisfy relation [4.35].

  In this chapter, we will study the ingredients of the PVP, namely:

- the VP of external and internal efforts,
- and the VP of the quantities of acceleration.

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

In Chapter 6, we will see how the PVP allows one to obtain the equations governing the motion of systems of rigid bodies. These equations are called Lagrange's equations.

• The VP mentioned in the PVP [5.1] are defined with respect to a Galilean reference frame R_g. However, we sometimes need to consider the VP relative to a non-Galilean (or non-inertial) reference frame. This may be the case:

– in the proof for the VP [5.14] of inter-efforts between two rigid bodies. We calculate the VP in a reference frame attached to a rigid body, which is often not Galilean,

– in definition [7.79] of a perfect joint (given in Chapter 7). It is better for the perfect character to be defined relative to any reference frame, not only a Galilean one.

Thus, for generality, in the rest of this chapter we will study the VP relative to any reference frame R_1, which is not necessarily Galilean. Of course, in order to obtain the VP in a Galilean reference frame R_g, one has just to apply the results obtained in this chapter by making $R_1 = R_g$.

5.2. VP of efforts internal to each rigid body

PVP [5.1] involves the set of efforts exerted upon system $\mathcal{S}$, both external and internal (refer to section 3.4):

1. The *external* efforts, $\mathcal{F}_{\text{ext}\to\mathcal{S}}$, are exerted by the exterior of the system on the rigid bodies of the system.

2. The *internal* efforts $\mathcal{F}_{\text{int}\to\mathcal{S}}$ include

   (a) *the inter-efforts* between the rigid bodies in the system,

   (b) and the efforts within each solid body in the system (a solid body is a body that is not reduced to a particle).

The efforts within a solid rigid body are not the classical efforts known in rigid body mechanics. Their schematization arises from the continuum mechanics. In the framework of rigid body mechanics, the following principle is accepted, which makes it possible to ignore these efforts:

Principle of zeroness of the VP of efforts internal to a rigid body. $\forall$ reference frame, $\forall t$, $\forall$ rigid body S, $\forall$ virtual velocity fields satisfying [4.35], the VP of the internal efforts (VP that we will not seek to define) is *zero*. [5.2]

Consequently, we will only consider the following efforts in the sequel:

1. The *exterior* efforts $\mathcal{F}_{\text{ext}\to\mathcal{S}}$, exerted by the exterior of the system on the rigid bodies of the system.

2. And the *interior* efforts $\mathcal{F}_{\text{int}\to\mathcal{S}}$, which reduce to the *inter-efforts* between the rigid bodies of the system.

5.3. VP of efforts

The efforts considered in the sequel include all the efforts exerted on system $\mathcal{S}$, except for those efforts that are internal to each rigid body, which, according to principle [5.2], do not come into play in rigid body mechanics.

Definition. Consider

– a set of forces applied on system $\mathcal{S}$, denoted by $Forces_{\to\mathcal{S}}$. This set is generally made up of concentrated or distributed forces. Using the notations in section 3.3, concentrated forces includes the forces $\vec{F}^{(i)}(t)$ applied at points A_i, and distributed forces are represented by a mass force $\vec{f}(A,t)$ distributed in the region V occupied by $\mathcal{S}$,

– a reference frame R_1 and a VFF $V^*_{R_1\mathcal{S}}$ over $\mathcal{S}$.

The virtual power, with respect to R_1 and at instant t, of the set of forces $Forces_{\to\mathcal{S}}$ in the VVF $V^*_{R_1\mathcal{S}}$ is, by definition:

$$\mathscr{P}^*_{R_1}(Forces_{\to\mathcal{S}},t) \equiv \sum_i \vec{F}^{(i)}(t)\cdot\vec{V}^*_{R_1\mathcal{S}}(A_i) + \int_{A\in V} \vec{f}^V(A,t)\cdot\vec{V}^*_{R_1\mathcal{S}}(A)\mathrm{d}m$$

Considering that a concentrated force is a special distributed force, the VP, with respect to R_1 and at instant t, of the forces in the VVF $V^*_{R_1\mathcal{S}}$, can be expressed in the following shortened form:

$$\boxed{\mathscr{P}^*_{R_1}(Forces_{\to\mathcal{S}},t) \equiv \int_{\mathcal{S}} \vec{f}(A,t)\cdot\vec{V}^*_{R_1\mathcal{S}}(A)\mathrm{d}m} \qquad [5.3]$$

To define the VP of torques, we restrict ourselves to a single rigid body and work with the VFFs *that satisfy* [4.35] on this rigid body.

Definition. Consider

– a set of torques applied on *a rigid body* S, denoted by $Torques_{\to S}$. This set is generally made up of concentrated or distributed torques. With the notations in section 3.3, the concentrated torques consist of torques $\vec{C}^{(i)}(t)$, the distributed torques are represented by a mass torque $\vec{c}(A,t)$, distributed in the region V occupied by $\mathcal{S}$,

– a VVF $\vec{V}^*_{R_1S}$, which *satisfies* [4.35] on S. We denote $\vec{\Omega}^*_{R_1R_S}$ the virtual angular velocity (or *one* virtual angular velocity, if the rigid body S is rectilinear).

The virtual power, with respect to R_1 and at instant t, of the set of torques $Torques_{\to S}$ in the VVF $V^*_{R_1S}$ is, by definition

$$\mathscr{P}^*_{R_1}(Torques_{\to S},t) \equiv \left(\sum_i \vec{C}^{(i)}(t) + \int_{A\in V} \vec{c}(A,t)\mathrm{d}m\right).\vec{\Omega}^*_{R_1R_S}$$

By considering that a concentrated torque is a special distributed torque, we can express the VP, with respect to R_1 and at instant t, of the torques in the VVF $\vec{V}^*_{R_1S}$ in the shortened form:

$$\boxed{\mathscr{P}^*_{R_1}(Torques_{\to S},t) \equiv \int_S \vec{c}(A,t)\mathrm{d}m\cdot\vec{\Omega}^*_{R_1R_S} = \int_S \vec{c}(A,t)\cdot\vec{\Omega}^*_{R_1R_S}\mathrm{d}m} \qquad [5.4]$$

REMARK. If the rigid body S is rectilinear, the virtual angular velocity $\vec{\Omega}^*_{R_1R_S}$ is determined within a component parallel to the rigid body. It is for this reason that in the statement, we spoke of *one* virtual angular velocity $\vec{\Omega}^*_{R_1R_S}$ in the case of a rectilinear rigid body.

Furthermore, a rectilinear rigid body S cannot undergo torques that are collinear to it. Thus, the torques must be orthogonal to S. Therefore, the indeterminate part of $\vec{\Omega}^*_{R_1R_S}$ does not come into play in the VP [5.4] and the VP under consideration is indeed well determined. ■

5.4. VP of efforts exerted on a rigid body

5.4.1. *General expression*

The following theorem shows that, like real power, the VP of efforts exerted on a *rigid body* can be expressed in a special form.

Theorem. Consider

- a *rigid body* S defining a reference frame R_S, a system of efforts $\mathcal{F}_{\to S}$ applied on S and made up of forces or torques that are either concentrated or distributed. The moment field of the efforts is $\mathcal{M}_{\to S}(t)$ of resultant $\vec{R}_{\to S}(t)$ (see section 3.6),
- a VVF $V^*_{R_1 S}$, which satisfies [4.35] on S. The virtual angular velocity (or *one* virtual angular velocity, if the rigid body S is rectilinear) is $\vec{\Omega}^*_{R_1 R_S}$.

The VP, with respect to R_1 and at instant t, of the effort system $\mathcal{F}_{\to S}$ in the VVF $V^*_{R_1 S}$ is equal to the product (symbolized by $\circ$) of the moment field of the efforts and the VVF:

$$\boxed{\mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t) = \vec{R}_{\to S}(t) \cdot \vec{V}^*_{R_1 S}(A) + \vec{\mathcal{M}}_{\to S}(A, t) \cdot \vec{\Omega}^*_{R_1 R_S} = \mathcal{M}_{\to S}(t) \circ V^*_{R_1 S}} \qquad [5.5]$$

PROOF. Starting from definition [5.3] and [5.4], we have

$$\mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t) = \int_S \vec{f}(B, t) \cdot \vec{V}^*_{R_1 S}(B) \mathrm{d}m + \int_S \vec{c}(B, t) \mathrm{d}m \cdot \vec{\Omega}^*_{R_1 R_S}$$

At the instant t considered, choosing a given point A and making use of relationship [4.35] – $\vec{V}^*_{R_1 S}(B) = \vec{V}^*_{R_1 S}(A) + \vec{\Omega}^*_{R_1 R_S} \times \overrightarrow{AB}$ – lead to

$$\mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t) = \underbrace{\int_S \vec{f}(B, t) \mathrm{d}m}_{\vec{R}_{\to S}} \cdot \vec{V}^*_{R_1 S}(A) + \underbrace{\left(\int_S \overrightarrow{AB} \times \vec{f}(B, t) \mathrm{d}m + \int_S \vec{c}(B, t) \mathrm{d}m \right)}_{\vec{\mathcal{M}}_{\to S}(A, t)} \cdot \vec{\Omega}^*_{R_1 R_S} \quad \blacksquare$$

Thus, if two effort systems applied to the same rigid body yield the same moment field, then they have the same VP.

5.4.2. *VP of zero moment field efforts exerted upon a rigid body*

The following result follows immediately from theorem [5.5]:

Corollary. Let S be *a rigid body* subjected to efforts $\mathcal{F}_{\to S}$ whose moment field is *zero*: $\vec{\mathcal{M}}_{\to S}(t) = 0, \forall t$. We have

$$\forall \text{ reference frame } R_1, \forall t, \boxed{\mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t) = 0} \qquad [5.6]$$

5.4.3. *Dependence of the VP of efforts on the reference frame*

Theorem. Let S be a rigid body subjected to efforts $\mathcal{F}_{\to S}$ (forces or torques, concentrated or distributed) whose moment field is $\mathcal{M}_{\to S}(t)$. We have

$$\forall t, \forall \text{ reference frames } R_1, R_2, \boxed{\mathscr{P}^*_{R_2}(\mathcal{F}_{\to S}, t) = \mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t) + \mathcal{M}_{\to S}(t) \circ \underbrace{(V^*_{R_2 S} - V^*_{R_1 S})}_{V^*_{R_2 S(R_1)}}} \qquad [5.7]$$

PROOF. On writing relationship [5.5] in the reference frames R_1 and R_2 respectively, and by then subtracting the relationships obtained, we arrive at

$$\mathscr{P}^*_{R_2}(\mathcal{F}_{\to S}, t) = \mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t) + \vec{\mathcal{M}}_{\to S}(t) \circ (V^*_{R_2 S} - V^*_{R_1 S})$$

According to [4.49], we have $V^*_{R_2 S} - V^*_{R_1 S} = V^*_{R_2 S(R_1)}$. ■

Theorem. [5.8]

The VP $\mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t)$ has *the same expression* for all reference frames R_1 that verify the following hypothesis:

HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and the point O_1 fixed in R_1 do *not* depend on q.

The expression for the VP does *not* depend on R_1. Thus, the VP will be written *without the reference frame index*: $\boxed{\mathscr{P}^*(\mathcal{F}_{\to S}, t)}$.

PROOF. The adopted hypothesis enables us to apply theorem [4.18]: the virtual velocity $\vec{V}^*_{R_1}(p)$ of the current particle p does not depend on R_1. Consequently, the VP $\mathscr{P}^*_{R_1}(\mathcal{F}_{\to S}, t)$ given by [5.5] does not depend on R_1 either. ■

If we also adopt hypothesis [2.33] for R_2, then the VP $\mathscr{P}^*_{R_2}(\mathcal{F}_{\to S}, t)$ with respect to R_2 does not depend on R_2. By inserting this in [5.7], we get $\mathcal{M}_{\to S}(t) \circ V^*_{R_2 S(R_1)} = 0$, which is predictable, given that we know from [4.44] that $V^*_{R_2 S(R_1)} = 0$.

5.5. VP of efforts exerted on a system of rigid bodies

The results obtained in the previous section for a rigid body can easily be generalized to the case of a system formed of several rigid bodies (some of which may be reduced to particles).

5.5.1. *General expression*

Theorem. Consider

- a system of efforts (forces and torques) $\mathcal{F}_{\to \mathcal{S}}$ applied to a system $\mathcal{S} = \bigcup_s S_s$ that is made up of several rigid bodies S_s. These efforts may be external and/or inter-efforts for $\mathcal{S}$. The moment field of the efforts applied to a rigid body S_s is $\mathcal{M}_{\to S_s}(t)$,
- a VVF $V^*_{R_1 \mathcal{S}}$ that *satisfies* [4.35] *on each rigid body* S_s (see [4.39]). The restriction of this VVF on each rigid body S_s is $V^*_{R_1 S_s}$, the virtual angular velocity (or one virtual angular velocity, if the rigid body S_s is rectilinear) is $\vec{\Omega}^*_{R_1 S_s}$.

The VP, with respect to R_1 and at instant t, of the effort system $\mathcal{F}_{\to \mathcal{S}}$ in the VVF $V^*_{R_1 \mathcal{S}}$ is

$$\boxed{\mathscr{P}^*_{R_1}(\mathcal{F}_{\to \mathcal{S}}) = \sum_s \mathcal{M}_{\to S_s}(t) \circ V^*_{R_1 S_s}} \qquad [5.9]$$

PROOF. The VP of the effort system is the sum of the VP of the efforts $\mathcal{F}_{\to S_s}$ on each rigid body S_s that makes up the system:

$$\mathscr{P}^*_{R_1}(\mathcal{F}_{\to \mathcal{S}}, t) = \sum_s \mathscr{P}^*_{R_1}(\mathcal{F}_{\to S_s}, t)$$

where the VP $\mathscr{P}^*_{R_1}(\mathcal{F}_{\to S_s}, t)$ on each rigid body number s is given by [5.5]. ■

5.5.2. *Dependence of the VP of the efforts on the reference frame*

Theorem. Let $\mathcal{S} = \underset{s}{\cup} S_s$ be a system composed of several rigid bodies S_s, subjected to an effort system $\mathcal{F}_{\to S}$ (forces or torques, concentrated or distributed) whose moment field is $\mathcal{M}_{\to S}(t)$.

The VP with respect to two distinct reference frames R_1 and R_2 and in two VVFs $V^*_{R_1\mathcal{S}}$ and $V^*_{R_2\mathcal{S}}$, respectively, are related through

$$\forall t, \forall \text{ reference frames } R_1, R_2, \boxed{\mathscr{P}^*_{R_2}(\mathcal{F}_{\to\mathcal{S}}, t) = \mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}, t) + \left(\sum_s \mathcal{M}_{\to S_s}(t)\right) \circ V^*_{R_2 S(R_1)}} \quad [5.10]$$

PROOF. On writing relationship [5.9] in the reference frames R_1 and R_2, and then subtracting the relationships obtained, we arrive at

$$\mathscr{P}^*_{R_2}(\mathcal{F}_{\to\mathcal{S}}, t) = \mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}, t) + \sum_s \left(\mathcal{M}_{\to S_s}(t) \circ (V^*_{R_2 S_s} - V^*_{R_1 S_s})\right)$$

According to [4.49], we can write $V^*_{R_2 S_s} - V^*_{R_1 S_s} = V^*_{R_2 S(R_1)}$. ■

The following result is obtained in the same manner as in the case of a single rigid body (theorem [5.8]):

Theorem. [5.11]

The VP $\mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}, t)$ has *the same expression* for all reference frames R_1 that verify the following hypothesis:

HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and the point O_1 fixed in R_1 do *not* depend on q.

The expression of the VP does *not* depend on R_1. Thus, the VP will be written *without the reference frame index*: $\boxed{\mathscr{P}^*(\mathcal{F}_{\to\mathcal{S}}, t)}$.

5.5.3. *VP of zero moment field efforts exerted on a system of rigid bodies*

The following result follows immediately from the previous theorem:

Corollary. [5.12]

We take the same data as in [5.9] and we additionally assume:

HYPOTHESIS: The moment field of the effort system $\mathcal{F}_{\to\mathcal{S}} \equiv \underset{s}{\cup} \mathcal{F}_{\to S_s}$ is *zero*: $\forall t, \sum_s \mathcal{M}_{\to S_s}(t) = 0$.

Thus, the VP in a reference frame of the effort system $\mathcal{F}_{\to S}$ exerted on $\mathcal{S}$ is *independent of this reference frame (but unlike the case with a rigid body, this VP may be non-zero)*.

The corollary [5.12] shows that in practice, in order to calculate the VP of the efforts whose moment field is zero, one had better work in a reference frame in which the calculations are simpler.

5.5.4. *VP of inter-efforts between the rigid bodies of a system*

Theorem. The VP, with respect to a reference frame R_1, of the inter-efforts between the rigid bodies in a system $\mathcal{S} = \underset{s}{\cup} S_s$ is *independent of* R_1. [5.13]

PROOF. As the sum of the moment fields of the inter-efforts between the rigid bodies of $\mathcal{S}$ is zero (see [3.18]), corollary [5.12] can be applied. ■

In [5.2], we postulated the zeroness of the VP of internal efforts of a rigid body. In the case of a system made up of several rigid bodies, the inter-efforts between the bodies in the system are efforts internal to the system. However, their VP is *not*, *a priori*, zero. The previous theorem only stipulates that their VP is independent of the reference frame relative to which it is calculated.

5.5.5. *The specific case of the inter-efforts between two rigid bodies*

Theorem. Let S_1 and S_2 bet two rigid bodies that exert, between themselves, the interaction efforts $\mathcal{F}_{S_1 \leftrightarrow S_2}$, made up of efforts $\mathcal{F}_{S_1 \to S_2}$ and $\mathcal{F}_{S_2 \to S_1}$, whose respective moment fields are $\mathcal{M}_{S_1 \to S_2}$ and $\mathcal{M}_{S_2 \to S_1} = -\mathcal{M}_{S_1 \to S_2}$ (which implies that the moment field of inter-efforts $\mathcal{M}_{S_1 \leftrightarrow S_2} = \mathcal{M}_{S_1 \to S_2} + \mathcal{M}_{S_2 \to S_1}$ is zero).

The VP (*independent of the reference frame*) of the inter-efforts between S_1 and S_2 is

$$\boxed{\mathscr{P}^*(\mathcal{F}_{S_1 \leftrightarrow S_2}, t) = \mathcal{M}_{S_2 \to S_1}(t) \circ V^*_{R_2 S_1}} \qquad [5.14]$$

where R_2 is the reference frame defined by the rigid body S_2. This VP is written *without the reference frame index* as it is independent of the reference frame with respect to which it is calculated.

FIRST PROOF. It results from $\mathcal{M}_{S_1 \leftrightarrow S_2} = 0$ and corollary [5.12] that the VP $\mathscr{P}^*(\mathcal{F}_{S_1 \leftrightarrow S_2}, t)$ is independent of the reference frame and that it may, thus, be calculated with respect to any reference frame. As it turns out that the calculation is simple in R_1 or R_2. We choose to apply [5.9] in one of these two reference frames, say R_2:

$$\mathscr{P}^*(\mathcal{F}_{S_1 \leftrightarrow S_2}, t) = \mathscr{P}^*_{R_2}(\mathcal{F}_{S_1 \leftrightarrow S_2}, t) = \mathcal{M}_{S_2 \to S_1}(t) \circ V^*_{R_2 S_1} + \mathcal{M}_{S_1 \to S_2}(t) \circ \underbrace{V^*_{R_2 S_2}}_{=0} \qquad ■$$

The proof shows that the efforts $\mathcal{F}_{S_1 \to S_2}$ have a zero VP with respect to R_2 and what remains is only the VP with respect to R_2 of the efforts $\mathcal{F}_{S_2 \to S_1}$.

SECOND PROOF. Given that the VP of the inter-efforts is independent of the reference frame, let us calculate this by placing ourselves in an arbitrary reference frame R_3 and by writing (t is omitted for brevity):

$$\begin{aligned}
\mathscr{P}^*(\mathcal{F}_{S_1 \leftrightarrow S_2}) &= \mathscr{P}^*_{R_3}(\mathcal{F}_{S_1 \leftrightarrow S_2}) \\
&= \mathcal{M}_{S_1 \to S_2} \circ V^*_{R_3 S_2} + \mathcal{M}_{S_2 \to S_1} \circ V^*_{R_3 S_1} && \text{according to [5.9]} \\
&= \mathcal{M}_{S_2 \to S_1} \circ (V^*_{R_3 S_1} - V^*_{R_3 S_2}) && \text{as } \mathcal{M}_{S_1 \to S_2} = -\mathcal{M}_{S_2 \to S_1} \\
&= \mathcal{M}_{S_2 \to S_1} \circ V^*_{R_2 S_1} && \text{using [4.49]} : V^*_{R_3 S_1} = V^*_{R_3 S(R_2)} + V^*_{R_2 S_1}
\end{aligned}$$

and by noting that $V^*_{R_3 S(R_2)} = V^*_{R_3 S_2}$ since the rigid body $S(R_2)$ defined by R_2 extends the rigid body S_2 to infinity (see [1.56] *sqq.*). ■

Observe that formula [4.49] for the composition of the VVF of rigid body has been used at different places:

– in the proofs of [5.7] and [5.10] to obtain $V^*_{R_2 S} = V^*_{R_2 S(R_1)} + V^*_{R_1 S}$, that is $V^*_{R_2 S(R_1)} = V^*_{R_2 S} - V^*_{R_1 S}$,

– in the second proof above to obtain $V^*_{R_3S_1} = V^*_{R_3S(R_2)} + V^*_{R_2S_1}$, that is $V^*_{R_2S_1} = V^*_{R_3S_1} - V^*_{R_3S(R_2)}$,

and that the expressions obtained do not all have the same form.

In theorem [5.14], the reference frame R_2 is defined by the rigid body S_2, which is one of the bodies in the studied system. Thus, the hypothesis in theorem [4.44] is *not* verified and we do *not* have $V^*_{R_2S_1} = 0$. The VVF $V^*_{R_2S_1}$ plays an essential role in the VP of inter-efforts between S_1 and S_2.

5.6. Summary of the cases where the VV and VP are independent of the reference frame

It would be helpful, at this point, to review the different cases where the VV and VP are independent of the reference frame with respect to which they are calculated.

1. For the VV, it has been seen that hypothesis [2.26] (which states that the rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q) makes it possible to simplify the expression for the VV of a particle and to arrive at expression [4.11] :

$$\vec{V}^*_{R_1}(p) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1P}}{\partial q_i} \dot{q}_i^*$$

However, hypothesis [2.26] is not sufficient for independence with respect to the reference frame. It was necessary to adopt another, stronger, hypothesis, namely

HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and the point O_1 fixed in R_1 does *not* depend on q, in order for the VV $\vec{V}^*_{R_1}(p)$ to be *independent of the reference frame* R_1. The VV of a particle is then written *without the reference frame index* and is given by [4.18]:

$$\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\partial \overrightarrow{P}}{\partial q_i} \dot{q}_i^*$$

2. As the VP is, *grosso modo*, the product of a force and a VV, hypothesis [2.33] has the same effect on the VPs as on the VVs. Using this hypothesis, it has been shown in [5.8] and [5.11] that the VP of efforts exerted on a rigid body or a system of rigid bodies is *independent of the reference frame* R_1 in which it is calculated. This makes it possible to write it *without the reference frame index*:

$$\mathscr{P}^*(\mathcal{F}_{\to S}, t) \text{ for a rigid body } S \qquad \text{or} \qquad \mathscr{P}^*(\mathcal{F}_{\to \mathcal{S}}, t) \text{ for a system of rigid bodies } \mathcal{S}$$

The Lagrange's equations will be established in later chapters taking R_1 equal to a Galilean reference frame R_g. Further, we will agree to choose $R_0 = R_g$, so that hypothesis [2.33] is *automatically satisfied*.

3. A hypothesis different from [2.33], namely that the moment field of efforts exerted is zero, also makes the VP independent of the reference frame. We have thus proved the following results:

   (a) Theorem [5.6]: If the moment field of efforts exerted on a rigid body is zero, the VP of these efforts is zero with respect to any reference frame.

(b) Theorem [5.11]: If the moment field of efforts exerted on a system of rigid bodies $\mathcal{S}$ is zero, the VP of these efforts is independent of the reference frame (but not necessarily zero). This VP is, thus, written *without the reference frame index*: $\mathscr{P}^*(\mathcal{F}_{\to\mathcal{S}}, t)$.

(c) Theorem [5.14]: In particular, the VP of inter-efforts between two solids S_1 and S_2 is independent of the reference frame. This VP is, thus, written *without the reference frame index*: $\mathscr{P}^*(\mathcal{F}_{S_1\leftrightarrow S_2}, t)$.

5.7. VP of efforts expressed as a linear form of the $\dot{q}_i^*$

In view of [5.9], it can be observed that the VP of efforts exerted on a system of rigid bodies is a linear form of the $\dot{q}_i^*$, as with the VVF. This property will be proven again, in a direct manner, in order to arrive at expression [5.16] below for the VP. This is the expression we will retain in Chapter 6 to form the right-hand side of Lagrange's equations.

Theorem and definition. Consider

– a system $\mathcal{S} = \bigcup_s S_s$ made up of one or more rigid bodies S_s,

– a system of efforts (forces and torques) $\mathcal{F}_{\to\mathcal{S}}$ applied to $\mathcal{S}$ and made up of forces or torques, concentrated or distributed in the mass. As in [5.3] and [5.4], we write, in shortened form, the forces using the symbol $\vec{f}$ and the torques using the symbol $\vec{c}$,

– a VVF $V^*_{R_1\mathcal{S}}$, which we know satisfies [4.35] on each rigid body S_s in $\mathcal{S}$. According to [4.21], the angular velocity vector $\vec{\Omega}^*_{R_1R_s}$ (R_s being the reference frame defined by the rigid body S_s) is

$$\forall s = 1, \ldots, n \qquad \vec{\Omega}^*_{R_1R_s} = \sum_{i=1}^{n} \vec{\omega}^i_{1s}\dot{q}_i^* \qquad [5.15]$$

As it was observed in section 2.7, some vectors $\vec{\omega}^i_{1s}$ in the previous expression may be zero.

Then the VP, with respect to R_1 and at instant t, of the effort system $\mathcal{F}_{\to\mathcal{S}}$ in the VVF $V^*_{R_1\mathcal{S}}$ takes the form

$$\boxed{\mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}) = \sum_{i=1}^{n} Q_i\dot{q}_i^*} \qquad [5.16]$$

where the coefficient Q_i, called *the ith generalized force associated with the efforts* $\mathcal{F}_{\to\mathcal{S}}$, is defined as

$$\boxed{Q_i \equiv \int_{\mathcal{S}} \vec{f}(P,t).\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)dm + \sum_s \int_{S_s} \vec{c}(P,t)dm\ .\vec{\omega}^i_{1s}} \qquad [5.17]$$

• If the following hypothesis is adopted:

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

Then, [5.17] becomes

$$\boxed{Q_i = \int_{\mathcal{S}} \vec{f}(P,t).\frac{\partial\overrightarrow{O_1P}}{\partial q_i}dm + \sum_s \int_{S_s} \vec{c}(P,t)dm\ .\vec{\omega}^i_{1s}} \qquad [5.18]$$

PROOF. According to definitions [5.3] and [5.4], we have

$$\mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}) = \int_{\mathcal{S}} \vec{f}(P,t)\cdot\vec{V}^*_{R_1}(p)\mathrm{d}m + \sum_s \int_{S_s} \vec{c}(P,t)\mathrm{d}m\cdot\vec{\Omega}^*_{R_1R_s}$$

where $\vec{V}^*_{R_1}(p)$ is given by [4.10] and $\vec{\Omega}^*_{R_1R_s}$ by [5.15]. Hence [5.17].

Using hypothesis [2.26], $\vec{V}^*_{R_1}(p)$ is given by [4.11] and we obtain [5.18]. ■

• Practical ways for calculating the generalized force Q_i:

– In certain simple cases, Q_i can directly be calculated using definition [5.17] or relationship [5.18].

– When the integration over S_s is complicated, it is better to calculate the VP of efforts using formula [5.9], and then to identify Q_i as *the coefficient of* $\dot{q}^*_i$.

• Let us recall that the efforts – apart from those internal to each rigid body – are classified in two equivalent ways:

1. either as *external efforts* and *internal efforts* according to definitions [3.2] and [3.3]:
   (a) the external efforts $\mathcal{F}_{\text{ext}\to\mathcal{S}}$ are exerted by the exterior of the system of rigid bodies on the system,
   (b) the internal efforts $\mathcal{F}_{\text{int}\to\mathcal{S}}$ comprise the *inter-efforts* between the rigid bodies of the system.
2. or as *given efforts* and *constraint efforts*, in accordance with hypothesis [3.9]. The constraint efforts include, on the one hand, the efforts between a rigid body of the system and the exterior and, on the other hand, the inter-efforts between the bodies of the system.

This classification is summarized in the double equality [3.11]:

$$\mathcal{F}_{\to\mathcal{S}} = \mathcal{F}_{\text{ext}\to\mathcal{S}} \cup \mathcal{F}_{\text{int}\to\mathcal{S}} = \mathcal{F}_{\text{given}\to\mathcal{S}} \cup \mathcal{F}_{\text{constraint}\to\mathcal{S}}$$

The VP of efforts [5.16] can thus be decomposed in two equivalent manners:

1. either $\mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}) = \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{ext}\to\mathcal{S}}) + \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{int}\to\mathcal{S}})$, according to $\mathcal{F}_{\to\mathcal{S}} = \mathcal{F}_{\text{ext}\to\mathcal{S}} \cup \mathcal{F}_{\text{int}\to\mathcal{S}}$,
2. or $\mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}) = \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{given}\to\mathcal{S}}) + \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{constraint}\to\mathcal{S}})$, according to $\mathcal{F}_{\to\mathcal{S}} = \mathcal{F}_{\text{given}\to\mathcal{S}} \cup \mathcal{F}_{\text{constraint}\to\mathcal{S}}$.

As a matter of fact, this second decomposition is the one that is preferred in analytical mechanics. We are thus led to the following definition:

Definition. The generalized forces Q_i are decomposed as follows

$$\forall i \in [1,n], \qquad \boxed{Q_i = D_i + L_i} \qquad [5.19]$$

In this expression:

– D_i is calculated by [5.17] or [5.18] but with the given efforts only, it is called *the ith generalized given force*.

– L_i is calculated using [5.17] or [5.18] but using constraint efforts only, it is called *the ith generalized constraint force*.

With this decomposition, the VP of efforts are written as the sum of two terms:

$$\mathscr{P}^*_{R_1}(\mathcal{F}_{\to\mathcal{S}}) = \sum_{i=1}^{n} Q_i \dot{q}_i^* = \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{given}\to\mathcal{S}}) + \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{constraint}\to\mathcal{S}})$$

with $\begin{cases} \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{given}\to\mathcal{S}}) = \displaystyle\sum_{i=1}^{n} D_i \dot{q}_i^* \\ \mathscr{P}^*_{R_1}(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \displaystyle\sum_{i=1}^{n} L_i \dot{q}_i^* \end{cases}$.

5.8. Potential

5.8.1. *Definition*

Below is a case where it is possible to give an explicit expression to part of the given generalized force D_i:

Definition. Consider a system of efforts $\mathcal{F}$ acting on the system $\mathcal{S}$ and possibly also on the exterior of $\mathcal{S}$ (for example, the pair of forces at the ends of a spring connecting a rigid body of the system and the exterior).

The system of efforts $\mathcal{F}$ is *derivable, in the reference frame* R_1*, from a potential* $\mathcal{V}_{R_1}$ (or *admits, in* R_1*, of a potential* $\mathcal{V}_{R_1}$), in the VVF $V^*_{R_1\mathcal{S}}$, if there exists a real function $\mathcal{V}_{R_1}(q,t)$ such that

$$\forall \text{ CVV } V^*_{R_1\mathcal{S}}, \qquad \boxed{\mathscr{P}^*_{R_1}(\mathcal{F}) = -\sum_{i=1}^{n} \frac{\partial \mathcal{V}_{R_1}}{\partial q_i} \dot{q}_i^*} \qquad [5.20]$$

In other words, the generalized forces corresponding to the system of efforts $\mathcal{F}$ are

$$\forall i \in [1,n], \qquad \boxed{Q_i(\mathcal{F}) = -\frac{\partial \mathcal{V}_{R_1}}{\partial q_i}} \qquad [5.21]$$

In general, efforts exerted on a system do not have a potential. When a potential exists, it is only defined within an additive time function $f(t)$.

In Newtonian mechanics, we use the concept of potential energy and, to a lesser extent, that of potential. The two concepts are quite close to one another and their definitions are both based on *real* velocities. In analytical mechanics, potential is a concept based on *the VVF*; it is similar, but is not always identical to the concept of potential energy used in Newtonian mechanics.

Efforts that are *a priori* unknown may develop a zero or non-zero VP depending on the choice of VVF. For instance, as will be seen in Chapter 7 devoted to perfect joints, the VP of certain constraint efforts is zero using one parameterization but non-zero in another. If the VP is zero, then there exists a potential that is constant. This shows that the existence of a potential *depends fundamentally on the choice of the VVF*.

5.8.2. *Examples of potential*

5.8.2.1. *Constant concentrated force*

Theorem. Consider a reference frame R_1 endowed with a coordinate system $(O_1; \vec{x}_1, \vec{y}_1, \vec{z}_1)$ and a concentrated force $\vec{F}(t)$ acting on a particle p whose position is P. The force $\vec{F}(t)$ may possibly depend on time, but will not depend on q.

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

The force $\vec{F}(t)$ is derivable in R_1 from the potential

$$\boxed{\mathcal{V}_{R_1}(q,t) = -\overrightarrow{O_1P}.\vec{F}(t) + const}, \qquad [5.22]$$

where $const$ denotes an arbitrary constant.

PROOF. Using hypothesis [2.26], we can apply [4.11] to calculate the VP with respect to R_1 of the force $\vec{F}(t)$:

$$\begin{aligned}\mathscr{P}^*_{R_1}(\vec{F}) &= \vec{F}(t).\vec{V}^*_{R_1}(p) = \vec{F}(t).\sum_{i=1}^{n}\frac{\partial\overrightarrow{O_1P}}{\partial q_i}\dot{q}_i^* \\ &= \sum_{i=1}^{n}\frac{\partial(\overrightarrow{O_1P}.\vec{F}(t))}{\partial q_i}\dot{q}_i^* \qquad \text{as } \vec{F}(t) \text{ does not depend on } q. \qquad \blacksquare\end{aligned}$$

REMARK. Hypothesis [2.26] is indispensable to get [5.23]. Without this hypothesis, according to [4.10] we would have:

$$\mathscr{P}^*_{R_1}(\vec{F}) = \vec{F}(t).\vec{V}^*_{R_1}(p) = \vec{F}(t).\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{01}.\overrightarrow{O_1P}\right)\dot{q}_i^*$$

Since $\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{01}(q,t)$, it is, *a priori*, impossible to find a function $\mathcal{V}_{R_1}(q,t)$ for the last side to be recast as $\sum_{i=1}^{n}\frac{\partial\mathcal{V}_{R_1}(q,t)}{\partial q_i}\dot{q}_i^*$. $\blacksquare$

5.8.2.2. *Field of constant forces*

The following result generalizes the previous result in the case of a distributed force.

Theorem. Let us consider a reference frame R_1 endowed with a coordinate system $(O_1;\vec{x}_1,\vec{y}_1,\vec{z}_1)$ and a system $\mathcal{S}$ of mass m and mass center G, which is subjected to a field of mass force $\vec{f}(t)$ that may possibly depend on time, but not on q.

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

The field of forces $\vec{f}(t)$ is derivable in R_1 from the potential

$$\boxed{\mathcal{V}_{R_1}(q,t) = -m\overrightarrow{O_1G}.\vec{f}(t) + const} \qquad [5.23]$$

PROOF. Using hypothesis [2.26], we may apply [4.40] and [4.42] to calculate the VP, in R_1, of the force field $\vec{f}(t)$:

$$\begin{aligned}\mathscr{P}^*_{R_1}(\vec{f}) &= \int_{\mathcal{S}}\vec{V}^*_{R_1}(p).\vec{f}(t)dm = \vec{f}(t).\int_{\mathcal{S}}\vec{V}^*_{R_1}(p)dm = m\vec{f}(t).\vec{V}^*_{R_1\mathcal{S}}(G) \\ &= m\vec{f}(t).\sum_{i=1}^{n}\frac{\partial\overrightarrow{O_1G}}{\partial q_i}\dot{q}_i^* \\ &= \sum_{i=1}^{n}\frac{\partial(m\overrightarrow{O_1G}.\vec{f}(t))}{\partial q_i}\dot{q}_i^* \qquad \text{as } \vec{f}(t) \text{ does not depend on} q. \qquad \blacksquare\end{aligned}$$

REMARK. Hypothesis [2.26] is indispensable to get [5.23]. Without this hypothesis, according to [4.40] and [4.41] we would arrive at

$$\begin{aligned}\mathscr{P}^*_{R_1}(\vec{f}) &= \int_{\mathcal{S}} \vec{V}^*_{R_1}(p).\vec{f}(t)dm = \vec{f}(t).\int_{\mathcal{S}} \vec{V}^*_{R_1}(p)dm = m\vec{f}(t).\vec{V}^*_{R_1\mathcal{S}}(G)\\ &= m\vec{f}(t).\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{01}.\overrightarrow{O_1G}\right)\dot{q}_i^*\end{aligned}$$

Since $\bar{\bar{Q}}_{01} = \bar{\bar{Q}}_{01}(q,t)$, it is, *a priori*, impossible to find a function $\mathcal{V}_{R_1}(q,t)$ to recast the last term in the form $\sum_{i=1}^{n}\frac{\partial \mathcal{V}_{R_1}(q,t)}{\partial q_i}\dot{q}_i^*$. ■

5.8.2.3. *The gravity field*

The case of the gravity field can be treated as a particular form of theorem [5.23].

Corollary. Consider a reference frame R_1 endowed with a coordinate system $(O_1;\vec{x}_1,\vec{y}_1,\vec{z}_1)$ and a system $\mathcal{S}$ of mass m and mass center G, subjected to a gravity field $\vec{f} = -g\vec{z}_1$ where g is a constant scalar.

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

This gravity field is derivable in R_1 from the potential

$$\boxed{\mathcal{V}_{R_1} = mgz_G + const}, \qquad [5.24]$$

where $z_G \equiv \overrightarrow{O_1G}.\vec{z}_1$ is the elevation of mass center G relative to the coordinate system $(O_1;\vec{x}_1,\vec{y}_1,\vec{z}_1)$. We often choose the constant equal to zero, such that $\mathcal{V}_{R_1}$ is zero when the center G is located on the zero elevation.

5.8.2.4. *Restoring force on a particle*

Theorem. Consider a reference frame R_1, endowed with a coordinate system $(O_1;\vec{x}_1,\vec{y}_1,\vec{z}_1)$ and a particle p that moves along the axis $O_1\vec{x}_1$. The position of p in R_0 is denoted by $P = pos_{R_0}(p,t)$ and $\overrightarrow{O_1P} = x\vec{x}_1$. The position of p in R_1 is defined by the abscissa x. It is assumed that the particle is subjected to the force $\vec{F} = \vec{F}(x,t) = -k(t)x\vec{x}_1$ where $k(t)$ is a given function.

The force $\vec{F}$ is derivable in R_1 from the potential

$$\boxed{\mathcal{V}_{R_1}(x,t) = \frac{1}{2}k(t)x^2 + const} \qquad [5.25]$$

PROOF. Definition [5.17] gives

$$\begin{aligned}Q_x &= \vec{F}.\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right) = -k(t)x\,\vec{x}_1.\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(x\,\bar{\bar{Q}}_{10}.\vec{x}_1\right)\\ &= -k(t)x\left(\bar{\bar{Q}}_{10}.\vec{x}_1\right).\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(x\,\bar{\bar{Q}}_{10}.\vec{x}_1\right) \qquad \text{on applying [A1.4] :}\\ &\qquad\qquad \forall\vec{a},\vec{b},\ \vec{a}.\bar{\bar{Q}}_{01}.\vec{b} = \vec{b}.\bar{\bar{Q}}^T_{01}.\vec{a} = \vec{b}.\bar{\bar{Q}}_{10}.\vec{a}\\ &= -k(t)x\,\vec{e}_1.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}(x\,\vec{e}_1) \qquad \text{on applying [1.31] : } \bar{\bar{Q}}_{10}.\vec{x}_1 = \vec{e}_1\\ &= -k(t)x \qquad \text{because } \sum_{i=1}^{n}\frac{\partial}{\partial q_i} \text{ reduces to } \frac{\partial}{\partial x} \qquad ■\end{aligned}$$

5.8.2.5. *Force due to a spring*

We will show that the efforts due to springs, which are given efforts, as was stated in section 3.5, admit of a potential. Consider a rigid body S_2 connected to another rigid body S_1 by a linear spring whose mass is negligible (Figure 5.1). The attachment points where the spring is connected to S_1 and S_2 are A and B, respectively. The efforts $\mathcal{F}_{1\to 2}$ exerted upon S_2 consist of a concentrated force at B, directed along $\overrightarrow{BA}$: $\boxed{\vec{F} = -k(\ell - \ell_0)\vec{i}}$, where k is a positive constant (called the *stiffness of the spring*), ℓ_0 and ℓ are, respectively, the unstretched length and the current length of the spring and $\vec{i}$ is the unit vector parallel to the spring, whose direction is from A to B.

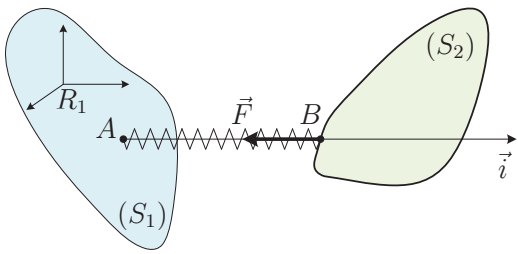


Figure 5.1. *Force exerted by a spring on a rigid body*

Theorem.

HYPOTHESIS:

(i) The attachment point A is attached to S_1.
(ii) The attachment point B is attached to S_2. [5.26]

Then, the force exerted by the spring on S_2 is derivable, *in the reference frame* R_1 *defined by* S_1, from the potential:

$$\boxed{\mathcal{V}_{R_1} = \frac{1}{2}k(\ell - \ell_0)^2 + const}, \qquad [5.27]$$

where the constant is often chosen to be equal to zero such that $\mathcal{V}_{R_1}$ is zero when the spring is unstretched.

FIRST PROOF. Definition [5.3] and relationship [5.5] give the VP for the force exerted by the spring on S_2

$$\mathscr{P}^*_{R_1}(\mathcal{F}_{1\to 2}) = \vec{F}.\vec{V}^*_{R_1 S_2}(B) \qquad [5.28]$$

Hypothesis $[5.26]_b$, combined with [1.58], entails that the particle b defined by $pos_{R_0}(b,t) = B, \forall t$ is a particle of S_2, hence $\vec{V}^*_{R_1 S_2}(B) = \vec{V}^*_{R_1}(b,t)$. Then, according to definition [4.10] and hypothesis $[5.26]_a$:

$$\vec{V}^*_{R_1 S_2}(B) = \vec{V}^*_{R_1}(b,t) = \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{AB}\right)\dot{q}_i^* \qquad [5.29]$$

With $\vec{F} = -k(\ell - \ell_0)\vec{i}$ and $\overrightarrow{AB} = \ell\vec{i}$, the VP [5.28] can be written as

$$\begin{aligned}
\mathscr{P}^*_{R_1}(\mathcal{F}_{1\to 2}) &= -k(\ell-\ell_0)\vec{i}.\bar{\bar{Q}}_{01}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\ell\vec{i}\right)\dot{q}_i^* \\
&= -k(\ell-\ell_0)\left(\bar{\bar{Q}}_{10}.\vec{i}\right).\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\ell\bar{\bar{Q}}_{10}.\vec{i}\right)\dot{q}_i^* \qquad \text{on applying [A1.4] :} \\
&\qquad\qquad \forall\vec{a},\vec{b},\ \vec{a}.\bar{\bar{Q}}_{01}.\vec{b} = \vec{b}.\bar{\bar{Q}}^T_{01}.\vec{a} = \vec{b}.\bar{\bar{Q}}_{10}.\vec{a} \\
&= -k(\ell-\ell_0)\vec{i}^{(1)}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\ell\vec{i}^{(1)}\right)\dot{q}_i^* \qquad \text{by writing } \vec{i}^{(1)} = \bar{\bar{Q}}_{10}.\vec{i} \\
&= -k(\ell-\ell_0)\vec{i}^{(1)}.\sum_{i=1}^{n}\left[\ell\frac{\partial\vec{i}^{(1)}}{\partial q_i} + \frac{\partial\ell}{\partial q_i}\vec{i}^{(1)}\right]\dot{q}_i^* \qquad [5.30]
\end{aligned}$$

as $\dfrac{\partial(\ell\vec{i}^{(1)})}{\partial q_i} = \ell\dfrac{\partial\vec{i}^{(1)}}{\partial q_i} + \dfrac{\partial\ell}{\partial q_i}\vec{i}^{(1)}$. Furthermore, by differentiating the equality $\vec{i}^{(1)}.\vec{i}^{(1)} = \vec{i}.\vec{i} = 1$ with respect to q_i, one finds $\vec{i}^{(1)}.\dfrac{\partial\vec{i}^{(1)}}{\partial q_i} = 0$. Hence

$$\mathscr{P}^*_{R_1}(\mathcal{F}_{1\to 2}) = -k(\ell-\ell_0)\sum_{i=1}^{n}\frac{\partial\ell}{\partial q_i}\dot{q}_i^* = -\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left[\frac{1}{2}k(\ell-\ell_0)^2\right]\dot{q}_i^* \qquad \blacksquare \qquad [5.31]$$

SECOND PROOF. The above proof is a little long as it works on the general expression [4.10] for the VV $\vec{V}^*_{R_1S_2}(B)$. It can be simplified a little if we adopt hypothesis [2.26], namely that the rotation tensor $\bar{\bar{Q}}_{01}$ does *not* depend on q. This hypothesis is restrictive; it is satisfied, in particular, if S_1 is a support fixed in R_0.

Using this hypothesis, the VV $\vec{V}^*_{R_1S_2}(B)$ is given by [4.11] and relationship [5.29] is replaced by $\vec{V}^*_{R_1S_2}(B) = \vec{V}^*_{R_1}(b,t) = \displaystyle\sum_{i=1}^{n}\frac{\partial\overrightarrow{AB}}{\partial q_i}\dot{q}_i^*$.

Relationship [5.30] becomes simpler: $\mathscr{P}^*_{R_1}(\mathcal{F}_{1\to 2}) = -k(\ell-\ell_0)\vec{i}.\displaystyle\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\ell\vec{i}\right)\dot{q}_i^*$, where $\dfrac{\partial(\ell\vec{i})}{\partial q_i} = \ell\dfrac{\partial\vec{i}}{\partial q_i} + \dfrac{\partial\ell}{\partial q_i}\vec{i}$.

Furthermore, by differentiating the equality $\vec{i}.\vec{i} = 1$ with respect to q_i, we obtain $\vec{i}.\dfrac{\partial\vec{i}}{\partial q_i} = 0$. We thus arrive at the same expression [5.31]. $\blacksquare$

5.8.2.6. *Inter-efforts due to a spring*

The spring considered in the above example exerts an external force on a rigid body. We now consider a spring connecting two rigid bodies belonging to the same system, such that the forces due to the spring are the *inter-efforts* between these two bodies. We will show that these inter-efforts also admit of a potential.

According to [5.14], the VP of the interaction efforts between two rigid bodies S_1 and S_2 is independent of the reference frame. Consequently, if the potential of the inter-efforts exists, it is also independent of the reference frame and it can be denoted by $\mathcal{V}$ *without the reference frame index*.

Moreover, it was assumed that the position of the system $S_1 \bigcup S_2$ in a reference frame R_1 depends on the position parameters q. The fact that potential $\mathcal{V}$ does not depend on R_1 suggests

that it is a function of position q via quantities, which do not depend on R_1 but only on *the shape* of $S_1 \bigcup S_2$ (that is, *the relative position* of S_1 and S_2). The reader can verify that this is indeed the case in the following examples.

• Consider two rigid bodies S_1, S_2 connected by a linear spring of zero mass. The attachment points of the spring on S_1 and S_2 are, respectively, A and B (Figure 5.2). The constraint inter-efforts include the efforts of S_1 on S_2 and the efforts of S_2 on S_1 via the spring. The respective moment field of these constraint inter-efforts are:

- $\mathcal{M}_{2\to1}$, whose resultant force is $\boxed{\vec{F}_{2\to1} = k(\ell - \ell_0)\vec{i}}$ (k is a positive constant) and moment at A is $\vec{\mathcal{M}}_{2\to1}(A) = \vec{0}$. ℓ_0 and ℓ are, respectively, the unstretched and the current length of the spring, $\vec{i}$ is the unit vector parallel to the spring, which points from A to B.

- $\mathcal{M}_{1\to2} = -\mathcal{M}_{2\to1}$, whose resultant force is $\boxed{\vec{F}_{1\to2} = -k(\ell - \ell_0)\vec{i}}$ and moment at B is $\vec{\mathcal{M}}_{1\to2}(B) = \vec{0}$. The last moment also is the moment at A as the resultant $\vec{F}_{1\to2}$ passes through A, so that we indeed have $\mathcal{M}_{1\to2} + \mathcal{M}_{2\to1} = 0$.

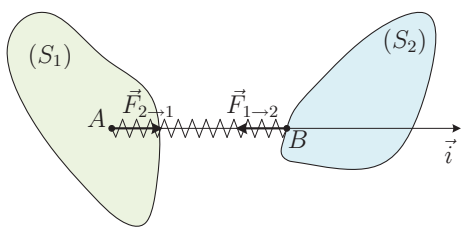


Figure 5.2. *Constraint inter-efforts due to a spring*

Theorem.

HYPOTHESIS: The attachment points A and B are *attached*, respectively, to the rigid bodies S_1 and S_2.

Then, the inter-efforts between S_1 and S_2 due to the spring are derivable (in any reference frame) from the potential:

$$\boxed{\mathcal{V} = \frac{1}{2}k(\ell - \ell_0)^2 + const} \qquad [5.32]$$

where the constant is often chosen to be equal to zero, so that $\mathcal{V}$ is zero when the spring is unstretched.

PROOF. On applying relationship [5.14], we can obtain the power (with respect to any reference frame) of the inter-efforts between S_1 and S_2 due to the spring

$$\begin{aligned} \mathscr{P}^*(\mathcal{F}_{1\leftrightarrow2}) &= \vec{F}_{1\to2}.\vec{V}^*_{R_1S_2}(B) + \vec{\mathcal{M}}_{1\to2}(B).\vec{\Omega}^*_{R_1R_2} \\ &= \vec{F}_{1\to2}.\vec{V}^*_{R_1S_2}(B), \end{aligned} \qquad [5.33]$$

where R_1, R_2 are the reference frames defined by S_1, S_2, respectively.

The rest of the proof is identical to that for [5.28]. ■

• We obtain a similar result for a *torsion* (or *spiral*) spring.

Theorem. Consider two rigid bodies S_1 and S_2 connected by a *torsion* spring. The relative motion of S_1 and S_2 is a rotation around a common axis oriented by a unit vector $\vec{i}$. The relative rotation is $\theta\vec{i}$, such that when $\theta = 0$ the spring is unstretched. The moment fields of the inter-efforts between the two rigid bodies are:

- $\mathcal{M}_{2\to1}$, whose resultant is $\vec{R}_{2\to1} = \vec{0}$ and moment at any point A is $\boxed{\vec{\mathcal{M}}_{2\to1}(A) = c\theta\vec{i}}$, where c is a positive constant (called the torsion stiffness of the spring),
- $\mathcal{M}_{1\to2} = -\mathcal{M}_{2\to1}$, whose resultant is $\vec{R}_{1\to2} = \vec{0}$ and moment at any point A is $\boxed{\vec{\mathcal{M}}_{1\to2}(A) = -c\theta\vec{i}}$.

Then, the inter-efforts between S_1 and S_2 due to the torsion spring are derivable from the potential

$$\boxed{\mathcal{V} = \frac{1}{2}c\theta^2 + const}, \qquad [5.34]$$

where the constant is often chosen to be equal to zero for V to be zero when the spring is unstretched.

PROOF. One has, at any point A

$$\begin{aligned}\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow2}) &= \vec{F}_{1\to2}.\vec{V}^*_{R_1S_2}(A) + \vec{\mathcal{M}}_{1\to2}(A).\vec{\Omega}^*_{R_1R_2} \text{ according to [5.14]} \\ &= \vec{\mathcal{M}}_{1\to2}(A).\vec{\Omega}^*_{R_1R_2}\end{aligned} \qquad [5.35]$$

Let us calculate the virtual angular velocity $\vec{\Omega}^*_{R_1R_2}$ using definition [4.21]. The relative (real) angular velocity between S_1 and S_2 is $\vec{\Omega}_{R_1R_2} = \dot{\theta}\vec{i}$, where the relative angle θ and the vector $\vec{i}$ depend, *a priori*, on the position parameters q and possibly on time t: $\theta = \theta(q,t)$, $\vec{i} = \vec{i}(q,t)$ (just as with the length ℓ of the linear spring). We thus have

$$\dot{\theta} = \sum_{i=1}^{n} \frac{\partial\theta}{\partial q_i}\dot{q}_i + \frac{\partial\theta}{\partial t}$$

Hence

$$\vec{\Omega}_{R_1R_2} = \sum_{i=1}^{n} \frac{\partial\theta}{\partial q_i}\vec{i}(q,t)\dot{q}_i + \frac{\partial\theta}{\partial t}\vec{i}(q,t)$$

By comparing this relationship with [2.35] or [2.46], we obtain $\vec{\omega}^i(q,t) = \dfrac{\partial\theta}{\partial q_i}\vec{i}(q,t)$ and $\vec{\omega}^t = \dfrac{\partial\theta}{\partial t}\vec{i}(q,t)$. Hence, using definition [4.21]

$$\vec{\Omega}^*_{R_1R_2} = \sum_{i=1}^{n} \vec{\omega}_i\dot{q}^*_i = \sum_{i=1}^{n} \frac{\partial\theta}{\partial q_i}\vec{i}(q,t)\dot{q}^*_i$$

(In particular, if one of the parameters q_i is equal to θ, we find $\vec{\Omega}^*_{R_1R_2} = \dot{\theta}^*\vec{i}$!) Finally, relationship [5.35] can be written as

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow2}) = -c\theta\sum_{i=1}^{n} \frac{\partial\theta}{\partial q_i}\dot{q}^*_i = -\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\frac{1}{2}c\theta^2\right)\dot{q}^*_i \qquad ■$$

• The following result extends [5.32] to the case of a *nonlinear* spring.

Theorem. Let S_1 and S_2 be two rigid bodies connected by a spring, as in Theorem [5.32], except that this time the spring is *nonlinear*. The force exerted by S_1 on S_2 is given by $\boxed{\vec{F}_{1\to 2} = -k(\lambda)\vec{i}}$, where $k(\lambda)$ is a known function (not necessary linear) of the spring elongation $\lambda \doteq \ell - \ell_0$, having the same sign as λ.

Then, the inter-efforts between S_1 and S_2 due to the spring are derivable from the potential

$$\boxed{\mathcal{V} = \int k(\lambda) d\lambda} \qquad [5.36]$$

PROOF. Relationship [5.33], which was written for a linear spring, remains valid here

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{F}_{1\to 2}.\vec{V}^*_{R_1 S_2}(B)$$

The VV $\vec{V}^*_{R_1 S_2}(B)$ is still given by [5.29]. However, as the expression for the tensile-compressive force has changed, relationship [5.31] now becomes

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = -k(\lambda)\sum_{i=1}^{n}\frac{\partial \ell}{\partial q_i}\dot{q}_i^* = -k(\lambda)\sum_{i=1}^{n}\frac{\partial \lambda}{\partial q_i}\dot{q}_i^* \qquad \blacksquare$$

5.9. VP of the quantities of acceleration

Quantity of acceleration was defined in section 1.13. Here, it is taken equal to $\vec{\Gamma}_{R_1\mathcal{S}}(P,t)dm = \rho\vec{\Gamma}_{R_1\mathcal{S}}(P,t)d\Omega$, where dm (respectively, $d\Omega$) is the mass (respectively, the volume) of an infinitesimal element within the system $\mathcal{S}$ and ρ is the density of the system. Since its unit is kg m / s^2 = N, a quantity of acceleration can be thought of as a *fictitious* force. The virtual power of such a force is defined as follows:

Definition. At an instant t, in a VVF $\vec{V}^*_{R_1\mathcal{S}}$, *the VP of the quantities of acceleration with respect to R_1 of $\mathcal{S}$*, denoted by $\mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}})$, is defined as

$$\boxed{\mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}}) \equiv \int_{\mathcal{S}} \vec{\Gamma}_{R_1\mathcal{S}}(P,t)\cdot\vec{V}^*_{R_1\mathcal{S}}(P)\,\mathrm{d}m} \qquad [5.37]$$

Theorem.

HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 does *not* depend on q.

Then, the VP of the quantities of acceleration $\mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}})$, at an instant t and in a VVF $V^*_{R_1\mathcal{S}}$, takes the form

$$\boxed{\mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}}) = \sum_{i=1}^{n} C_i\dot{q}_i^*} \qquad [5.38]$$

where the coefficients C_i are

$$\boxed{C_i = \frac{d}{dt}\left(\frac{\partial E^c_{R_1\mathcal{S}}}{\partial \dot{q}_i}\right)(q(t),\dot{q}(t),t) - \frac{\partial E^c_{R_1\mathcal{S}}}{\partial q_i}(q(t),\dot{q}(t),t)}$$

or, in shortened form $\boxed{C_i = \frac{d}{dt}\left(\frac{\partial E^c_{R_1\mathcal{S}}}{\partial \dot{q}_i}\right) - \frac{\partial E^c_{R_1\mathcal{S}}}{\partial q_i}}$ [5.39]

and the parameterized kinetic energy $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$ is defined by [2.54].

PROOF. Using the adopted hypothesis, the VV of the particle p with respect to R_1 is given by [4.11]. Inserting this relationship into definition [5.37] gives

$$\mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}}) \equiv \int_{\mathcal{S}} \vec{\Gamma}_{R_1}(p,t)\cdot\vec{V}^*_{R_1}(p)\mathrm{d}m = \sum_{i=1}^{n}\int_{\mathcal{S}} \vec{\Gamma}_{R_1}(p,t)\cdot\frac{\partial\overrightarrow{O_1P}}{\partial q_i}\mathrm{d}m\,\dot{q}^*_i = \sum_{i=1}^{n} C_i\dot{q}^*_i$$

by denoting

$$\boxed{C_i \equiv \int_{\mathcal{S}} \vec{\Gamma}_{R_1}(p,t)\cdot\frac{\partial\overrightarrow{O_1P}}{\partial q_i}\mathrm{d}m} \qquad [5.40]$$

On applying the Lagrange kinematic formula [2.52], which was established using hypothesis [2.26], we have

$$C_i = \int_{\mathcal{S}}\left[\frac{d}{dt}\frac{\partial\left(\frac{1}{2}\vec{V}^2_{R_1}(q,\dot{q},t)\right)}{\partial\dot{q}_i} - \frac{\partial\left(\frac{1}{2}\vec{V}^2_{R_1}(q,\dot{q},t)\right)}{\partial q_i}\right]\mathrm{d}m$$

Assuming that one can interchange integrals and derivatives with respect to $q_i, \dot{q}_i, t$, the above relationship becomes

$$\mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}}) = \sum_{i=1}^{n}\left[\frac{d}{dt}\left(\frac{\partial E^c_{R_1\mathcal{S}}}{\partial\dot{q}_i}\right) - \frac{\partial E^c_{R_1\mathcal{S}}}{\partial q_i}\right]\dot{q}^*_i \quad \blacksquare$$

If the coefficient $\dot{q}^*_i$ in [5.38] has the dimension of a velocity, the coefficient C_i has the dimension of a force and it may be called the *ith generalized quantity of acceleration*.

The coefficients C_i in [5.39] were obtained through application of the Lagrange kinematic formula [2.52]. Their expression is, therefore, governed by the same rules of calculation used in [2.52].

Note.

The following important points must be kept in mind:

1. In accordance with definition [2.43], $\vec{V}^2_{R_1}(q,\dot{q},t)$ (which appears in [2.54]) must not be treated as a composite function of t, but as a function of $2n+1$ *independent* variables. The expression for $\vec{V}^2_{R_1}$ is given by [2.53].

   Using definition [2.54], the parameterized kinetic energy, $E^c_{R_1\mathcal{S}}(q,\dot{q},t)$, must also be considered as a function of $2n+1$ *independent* variables. The expression for this parameterized kinetic energy is given by [2.55].

2. Thus, it is important to calculate coefficients C_i in [5.39] by carrying out the following operations *in order*:

   (a) Calculate the derivatives $\dfrac{\partial E^c_{R_1\mathcal{S}}}{\partial\dot{q}_i}$, $\dfrac{\partial E^c_{R_1\mathcal{S}}}{\partial q_i}$ with respect to $\dot{q}_i$, q_i, considering $\dot{q}_i$ and q_i to be *independent* variables.

   (b) Next, in the expressions obtained, replace q by $q(t)$ and $\dot{q}$ by the derivative $\dot{q}(t) = \frac{dq(t)}{dt}$.

   (c) Finally, calculate the derivative $\frac{d}{dt}$ as the derivative of the composite function of t thus formed.

3. It should be emphasized that the parameterized kinetic energy, $E^c_{R_1\mathcal{S}}(q, \dot{q}, t)$, must be considered as a function of $2n + 1$ *independent* variables $(q, \dot{q}, t)$. Thus, before deriving $E^c_{R_1\mathcal{S}}(q, \dot{q}, t)$ (the first operation listed above), we must take care *not* to use any complementary constraint equation that may exist in parameterization [2.19], or an equation obtained elsewhere (by Lagrange or others), in order to eliminate a q_i or a $\dot{q}_i$ from the expression of $E^c_{R_1\mathcal{S}}(q, \dot{q}, t)$.

6

Lagrange's Equations

In this chapter, we will establish the so-called Lagrange's equations for a system $\mathcal{S}$, made up of rigid bodies, some of which may be reduced to particles. This system is subjected to various mechanical efforts (given or constraint efforts, derivable or not from a potential) and to various mechanical joints (external or internal, holonomic or non-holonomic, perfect or not).

A *Galilean* reference frame R_g is assumed to be known. The Lagrange's equations will be obtained by means of the principle of virtual powers (PVPs) discussed in Chapter 5 and written in the reference frame R_g. As the PVP is equivalent to the Newton's laws, the Lagrange's equations are the equivalent of the equations arising from Newton's laws, rather than being complementary to them. Lagrange's equations are often combinations of equations derived from Newton's laws.

Lagrange's equations in the particular (and common) case of perfect joints will be examined in Chapter 8.

6.1. Choice of the common reference frame R_0

The common reference frame R_0 has been defined in [1.24] as a particular reference frame that is arbitrarily chosen, in which we report all the observed positions of particles (and, in addition, it is agreed that we denote the positions in R_0 without the index 0).

Up to now the reference frame R_0 has been freely chosen among all existing reference frames in the problem considered. In this chapter, we decide to adopt the following convention, which will determine the choice of R_0:

Convention on the choice of the common reference frame R_0. As the Galilean reference frame R_g is known, we choose the common reference R_0 equal to R_g:

$$\boxed{R_0 = R_g} \qquad [6.1]$$

This choice simplifies the presentation in the sequel since the rotation tensor of R_g with respect to R_0 is then equal to the identify tensor, $\bar{\bar{Q}}_{0g} = \bar{\bar{I}}$, with the result that the reference frames (R_g, R_0) *automatically satisfy both hypotheses* [2.26] *and* [2.33], which were seen several times in the previous chapters and which are stated once again in the following:

1. HYPOTHESIS [2.26]: The rotation tensor $\bar{\bar{Q}}_{0g}$ depends explicitly on time alone and *not* on q.

2. HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{0g}$ and the point O_g fixed in R_g do *not* depend on q. This hypothesis *includes* [2.26].

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

As was seen in [4.18], hypothesis [2.33], which is now satisfied, implies that the VV $\vec{V}^*_{R_g}(p)$ of a particle is, indeed, *independent of the reference frame* R_g. The VV is then written *without the reference frame index*:

$$\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i} \dot{q}_i^*$$

where $\frac{\overrightarrow{\partial P}}{\partial q_i}$ denotes $\frac{\partial \overrightarrow{O'P}}{\partial q_i}$, O' being any point independent of q, for example the origin O_g.

On the other hand, according to [5.8] and [5.11], hypothesis [2.33] implies that the VP of a system of efforts $\mathcal{F}_{\to S}$ exerted on the system $\mathcal{S}$ is *independent of the reference frame* R_g with respect to which it is calculated, which makes it possible to write it *without the reference frame index*: $\mathscr{P}^*(\mathcal{F}_{\to \mathcal{S}}, t)$.

In particular, the VP of the constraint efforts exerted by the mechanical joints on system $\mathcal{S}$ is *independent of the reference frame* R_g with respect to which it is calculated and it is denoted by $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to \mathcal{S}}, t)$.

The hypotheses [2.26] and [2.33] are required to establish the Lagrange's equations. By looking more closely at this, it can be seen that hypothesis [2.26] is sufficient to prove the Lagrange's equations [6.2] in a Galilean reference frame, whereas the stronger hypothesis [2.33] is required to prove the Lagrange's equations [6.19] in a non-Galilean reference frame.

REMARK. Let us add another comment on hypotheses [2.26] and [2.33].

– In the previous chapters, we differentiated between these hypotheses by choosing either one or the other, depending on the case, as the former was slightly weaker than the latter. This might be laborious, but necessary in order to prove the most general results possible for the real velocities, the VVs and the PVs. These results were general in the sense that they required the minimal hypothesis.

– From now onwards, we will move on to the applications of the previous chapters, namely, Lagrange's equations, the first integrals and the equilibrium. From the beginning, we decide to adopt convention [6.1], which is simple and includes both hypotheses. ■

6.2. Lagrange's equations

Consider a system $\mathcal{S}$ made up of one or more rigid bodies. Its *a priori* position in R_0 is described by parameterization [2.19] with n retained position parameters $q \equiv (q_1, \ldots, q_n)$. The position P of a current particle of the system, in R_0 and at the instant t, is $P = P(q, t)$ (see [2.21]).

We will describe the motion of the system relative to a reference frame R_g, which is assumed to be Galilean. The parameterized kinetic energy $E^c_{R_g\mathcal{S}}(q, \dot{q}, t)$ of the system $\mathcal{S}$ in R_g is defined by [2.54]. The system is subjected to:

– given efforts (for example, the weight of the system or the forces due to elastic springs), which yield the generalized forces $D_i, \in [1, n]$ (see definition [5.19]);

– and constraint efforts, which yield the generalized forces $L_i, \in [1, n]$.

The set of these efforts thus produces the generalized forces $Q_i = D_i + L_i, i \in [1, n]$.

Theorem.

HYPOTHESIS: The reference frame R_g is *Galilean* and we choose $R_0 = R_g$ according to convention [6.1].

We then have the following equations, called the *Lagrange's equations* of the system $\mathcal{S}$:

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{d}{dt}\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i} = Q_i = D_i + L_i} \qquad [6.2]$$

PROOF. The Lagrange's equations are obtained from the PVP [5.1].

– As the PVP holds for any VVF, it can be applied by restricting ourselves to VVFs $V^*_{R_g\mathcal{S}}$ given by [4.11]. We will, thus, apply the PVP [5.1] writing "$\forall$ VVFs $V^*_{R_g\mathcal{S}}$ given by [4.11]" instead of "$\forall$ VVFs".

– We admit that the considered set of VVFs $V^*_{R_g\mathcal{S}}$ can be obtained by assigning arbitrary values to the n-tuple $(\dot{q}^*_1, \ldots, \dot{q}^*_n)$. Thus, the PVP will be applied, writing "$\forall(\dot{q}^*_i)_{1\leq i\leq n}$" instead of "$\forall$ VVFs $V^*_{R_g\mathcal{S}}$ given by [4.11]".

Let us use results [5.2], [5.16] and [5.38] established in 5, recalling that hypothesis [2.26], which enabled us to prove [5.38], is satisfied here because of convention [6.1]. Bringing together these results, the PVP can be written as

$$\forall t, \forall(\dot{q}^*_i)_{1\leq i\leq n}, \qquad \sum_{i=1}^{n} C_i\dot{q}^*_i = \sum_{i=1}^{n} Q_i\dot{q}^*_i \qquad \text{i.e.} \sum_{i=1}^{n}(C_i - Q_i)\dot{q}^*_i = 0 \qquad [6.3]$$

Hence

$$\forall t, \forall i \in [1, n], \qquad C_i = Q_i \qquad \blacksquare$$

The Lagrange's equations obtained, as well as the number of these equations, are dependent on the parameterization used. Thus, in order to be precise, we should say that [6.2] are Lagrange's equations of system $\mathcal{S}$ *endowed with the parameterization* [2.19] instead of writing that they are the "Lagrange's equations of the system $\mathcal{S}$".

The Lagrange's equations [6.2] are general and hold in all cases: the chosen parameterization may be total, reduced or independent (see definition [2.18]), the constraint equations may be holonomic or not, the efforts may be given or unknown, and the mechanical joints may be perfect or not (the concept of a perfect joint will be defined in Chapter 7).

In accordance with what was seen in Chapter 5, we should comply with the following rules when establishing the Lagrange's equations:

1. The left-hand side $C_i \equiv \dfrac{d}{dt}\dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i}$ must be calculated according to the rules stipulated in section 5.9.

   The parameterized kinetic energy $E^c_{R_g\mathcal{S}}(q, \dot{q}, t)$ must be considered as a function of $2n+1$ *independent* variables $(q, \dot{q}, t)$.

   Further, we must *not* use any possible complementary constraint equation that exists in parameterization [2.19], nor any equations obtained elsewhere (through Lagrange or any other means) to replace a q_i or a $\dot{q}_i$ in $E^c_{R_g\mathcal{S}}$ by other position parameters. If we were to modify the expression of $E^c_{R_g\mathcal{S}}$ in this way, we would obtain *incorrect* Lagrange's equations.

   We must calculate the derivatives $\dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i}$, $\dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i}$ with respect to $\dot{q}_i$, q_i by considering these to be *independent* variables.

2. The generalized forces Q_i are relative to *all* the efforts exerted on the rigid bodies of the system, both the given efforts as well as the constraint efforts (the constraint efforts physically enforce the *primitive* and complementary constraint equations).

While complementary constraint equations cannot be used to modify the expression for $E^c_{R_g\mathcal{S}}(q,\dot{q},t)$ before calculating the derivatives $\dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i}$, $\dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i}$, they may freely be used *after* obtaining these derivatives.

• Let us specify the expression of the generalized force D_i in Lagrange's equations [6.2] by decomposing the given efforts into two categories:

– those that are derivable from a potential $\mathcal{V}_{R_g}$, which yield the generalized forces $-\dfrac{\partial \mathcal{V}_{R_g}}{\partial q_i}, i \in [1,n]$ (see definitions [5.20]–[5.21]);

– those that are not derivable from a potential, which yield the generalized forces denoted $D'_i, i \in [1,n]$.

This leads us to express the generalized forces corresponding to the given efforts as

$$\boxed{D_i = -\frac{\partial \mathcal{V}_{R_g}}{\partial q_i} + D'_i} \qquad [6.4]$$

whence the Lagrange's equations [6.2] become

$$\forall t, \forall i \in [1,n], \qquad \boxed{\frac{d}{dt}\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i} + \frac{\partial \mathcal{V}_{R_g}}{\partial q_i} = D'_i + L_i} \qquad [6.5]$$

As with the kinetic energy, the complementary constraint equations must *not* be used to modify the expression for potential $\mathcal{V}_{R_g}(q,t)$ before calculating the derivative $\dfrac{\partial \mathcal{V}_{R_g}}{\partial \dot{q}_i}$. These complementary constraint equations can only be used *after* differentiation.

Establishing Lagrange's equations thus requires the calculation of four ingredients:

1. the parameterized kinetic energy $E^c_{R_g\mathcal{S}}(q,\dot{q},t)$,
2. the potential $\mathcal{V}_{R_g}(q,t)$,
3. the generalized forces D'_i corresponding to given efforts that do not admit of a potential,
4. and the generalized forces L_i corresponding to the unknown constraint efforts.

The kinetic energy $E^c_{R_g\mathcal{S}}$ and the potential $\mathcal{V}_{R_g}$ are obtained as described in Chapter 5. Let us examine the generalized forces D'_i and L_i:

– Knowing the expression for the given efforts not derivable from a potential, their VP can be easily calculated and from this we can derive the D'_i coefficients.

– The calculation of the L_i coefficients is not so straightforward. Here, the existing mechanical joints must first be analyzed and the constraint efforts $\mathcal{F}_{\text{constraint}\to\mathcal{S}}$ acting on the system must be identified. We then calculate the VP of the constraint efforts and from this we derive the L_i coefficients. Generally speaking, the L_i coefficients depend on the constraint efforts. However, depending on the chosen parameterization, it may be that certain constraint efforts do not appear among the L_i coefficients.

The result of this operation can be summarized as follows:

$$\forall i \in [1,n], \qquad L_i = \text{ function of (all or certain) constrait efforts } \mathcal{F}_{\text{constraint}\to\mathcal{S}} \qquad [6.6]$$

There does exist a case where one does not have to study the joints in detail and where the calculation of the L_i coefficients becomes a systematic and simple procedure. This is the case of *perfect joints*, which will be studied in Chapters 7–8.

• Since the kinetic energy $E^c_{R_g\mathcal{S}}$ has the form [2.55] and the potential $\mathcal{V}_{R_g}$ is a function of (q, t), the left-hand-side term in [6.5] is a function of $(q(t), \dot{q}(t), \ddot{q}(t), t)$.

Consequently, the Lagrange's equations [6.5] yield n *second-order time differential equations*, which are, in general, *nonlinear and coupled*.

• It will be seen from the examples that the Lagrange's equations and the equations resulting from Newton's laws are equivalent in the sense that the Lagrange's equations are linear combinations of Newton's equations or vice versa.

6.3. Review and the need to model joints

Let us review the unknowns and the equations involved when studying the motion of the system $\mathcal{S}$.

If no constraint effort exists in the problem under consideration (in other words, if the only efforts that come into play are given efforts), then the generalized constraint forces $L_1, \ldots, L_n$ are zero. In this case, the Lagrange's equations [6.5] form n equations for n unknowns q and the problem is (theoretically) closed. This is what happens in celestial mechanics, where only gravitational force (the at-a-distance force given by the gravitational law) is present.

On the other hand, if we are examining a single rigid body that is connected to the exterior or several rigid bodies that are connected to each other and/or to the exterior, then constraint efforts are inevitable and the situation becomes more complex:

Summary of the equations and the unknowns.

(i) We have the following unknowns:

- the n position parameters $q \equiv (q_1, \ldots, q_n)$,
- and the constraint efforts present in expressions [6.6] for the generalized constraint forces L_i. Let us denote the number of such efforts by $n_{f\ell}$.

Thus, there are a total of $n + n_{f\ell}$ unknowns.

(ii) There are $n + \ell$ equations, which consist of

- exactly n Lagrange's equations [6.5] (no more and no less, with the parameterization used),
- ℓ complementary constraint equations whose number depends on the chosen parameterization. These equations, which may or may not be holonomic, are assumed to be *independent*.

In general, in a dynamic problem, it turns out that the number of complementary constraint equations is smaller than the number of constraint efforts present in [6.6]: $\boxed{\ell \leq n_{f\ell}}$.

It can be seen from this summary that in general $n_{f\ell} - \ell$ *equations are lacking*. Consequently, the mechanical principles alone – the PVP or Newton's laws – are not enough to solve motion problems. To obtain the lacking $n_{f\ell} - \ell$ relationships, we must study the physical nature of the

contacts at the mechanical joints and choose a *modelization of the joints* so as to obtain additional relationships, which are called the *contact laws*.

One particular model that is of great interest in mechanics is that of the *perfect joint*: in the case of perfect joints, the equality $\ell = n_{f\ell}$ occurs systematically and, as a consequence, the Lagrange's equations and complementary constraint equations are sufficient to yield as many equations as unknowns.

To summarize this: in the general case, there are $n + n_{f\ell}$ unknowns (n kinematic unknowns, q and $n_{f\ell}$ unknown constraint efforts) and $n + n_{f\ell}$ equations to be solved:

$$\begin{cases} \cdot\ n \text{ Lagrange's equations [6.5]}, \\ \cdot\ \ell \text{ complementary constraint equations}, \\ \cdot\ n_{f\ell} - \ell \text{ contact laws}. \end{cases} \qquad [6.7]$$

Furthermore, we may also have to verify some *inequalities* imposed by unilateral joints. For example, in the case of a point contact, the normal contact force N must satisfy the condition $N \geq 0$; if it is assumed that there is no slipping, we must also verify an inequality of the type $\|\vec{T}\| \leq fN$, where $\vec{T}$ is the tangential contact force and f is the coefficient of friction.

REMARKS.

1. For the system to be able to move, the number of constraint equations must be strictly smaller than the number n of the position parameters (the difference between the number of position parameters and the number of constraint equations gives the degrees of freedom of the system). As the number of complementary constraint equations is always smaller than the number of constraint equations, it follows that we must have $\boxed{\ell < n}$.

2. If all the constraint equations are holonomic and their number is greater than n, then we have a *static* problem (isostatic or hyperstatic), where the system does not move. Statics and equilibrium problems will be studied in Chapter 10.

3. To solve hyperstatic problems, where there is a deficit of equations, we must use the mechanics of *deformable* bodies. The principles of mechanics alone will not provide a sufficient number of *field* equations and we must use additional relationships, called *constitutive laws* for the materials being considered (laws that relate stresses to strains). ■

Definition. A function $t \mapsto q(t)$ of class C^2 is *a motion* of $\mathcal{S}$ if it satisfies all the equations [6.7], for a certain set of values for the existing constraint efforts. [6.8]

We will accept the principle that the *set of motions as defined in* [6.8] *is the set of all possible physical motions of* $\mathcal{S}$. In other words, any motion as defined in [6.8] in fact defines a possible physical motion of the mechanical system $\mathcal{S}$. Conversely, any possible physical motion of $\mathcal{S}$ is defined from a motion in [6.8].

6.4. Existence and uniqueness of the solution

The summary given above shows that there are as many equations as unknowns and we may, thus, hope to be able to solve the mechanical problem. The question that arises now is whether there does indeed exist a solution to the equations system [6.7] and, if so, whether the solution is unique.

• To respond to these questions, let us carry out a preliminary transformation that is not of much use from the mechanical point of view, but that is necessary for the mathematical analysis.

Given the explicit expression [2.55] for the parameterized kinetic energy $E^c_{R_g\mathcal{S}}(q,\dot{q},t)$, it can be observed that in the ith Lagrange's equation [6.5], the derivatives $\dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i}$, $\dfrac{\partial \mathcal{V}_{R_g}}{\partial q_i}$ and the generalized forces D'_i, L_i do not contain the second derivatives $\ddot{q}_j$. Only the derivative $\dfrac{d}{dt}\dfrac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i}$ contains the second derivatives $\ddot{q}_j$ via the term $\displaystyle\sum_{j=1}^{n} a_{ij}(q,t)\ddot{q}_j$.

Therefore, the Lagrange's equations [6.5] are second-order differential equations, linear with respect to the second derivatives $\ddot{q}_j$, and they take the form

$$\forall i \in [1,n], \qquad \sum_{j=1}^{n} a_{ij}(q,t)\ddot{q}_j + \text{ function of } (q,\dot{q},t) = L_i$$

and thus, by recasting these equations in matrix form:

$$[a(q,t)]\{\ddot{q}\} + \text{ column vector function of } (\{q\},\{\dot{q}\},t) = \{L\}$$

where $[a(q,t)]$ is the $n \times n$ square matrix whose (i,j) component is a_{ij}, $\{q\}$ (respectively, $\{L\}$) is the column vector whose ith component is q_i (respectively, L_i).

According to [2.56], the matrix $[a(q,t)]$ is symmetric. It is positive because

$$\forall \{\dot{q}\}, \qquad \{\dot{q}\}^T[a(q,t)]\{\dot{q}\} = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(q,t)\,\dot{q}_i\dot{q}_j = \int_{\mathcal{S}} \left(\sum_{i=1}^{n} \frac{\partial \overrightarrow{O_1P}}{\partial q_i}\right)^2 dm \geq 0$$

If we remove singular cases and retain only the so-called "regular" cases, where the matrix $[a(q,t)]$ is *positive definite*, this matrix is *invertible*.

Therefore, if there are no constraint efforts (i.e. if the L_i are zero), then the Lagrange's equations can be written in the form of n differential equations, *resolved* in the second derivatives $\ddot{q}$:

$$\boxed{\ddot{q} = F(q,\dot{q},t)} \qquad [6.9]$$

where, according to the hypotheses already adopted, the function F is continuous over its domain.

When the Lagrange's equations contain unknown constraint efforts, it is often possible to use the set of equations [6.7] in order to eliminate the constraint efforts and thus arrive at equations of the form [6.9].

The fact that the Lagrange's equations generally lead to equations of the form [6.9] is a very important point from the mathematical point of view.

• Once the form [6.9] is obtained, the problem of the existence and uniqueness of the solution is stated as follows:

– Take an initial instant t_0 and the initial conditions $q_0, \dot{q}_0$, such that $(t_0, q_0, \dot{q}_0)$ belongs to the domain of F.

– Is there a solution $t \mapsto q(t)$ for equation [6.9], defined on a time interval (open, semi-open or closed) containing t_0, and verifying $q(t_0) = q_0$ and $\dot{q}(t_0) = \dot{q}_0$? If so, is this solution unique?

The above-mentioned time interval may or may not be given. If it is not given, we seek to obtain the largest possible time interval.

Mathematical analyses of equation [6.9] have led to *sufficient* conditions for the existence and uniqueness of a local solution to the previous initial-value problem (Cauchy–Lipschitz conditions) and even for the extension of this local solution. We will retain only two important results:

– If there are no inequalities to be verified in addition to equations [6.7], then in the overwhelming majority of cases, *the above-stated initial-value problem has a unique solution* that can be extended in a more or less large time interval, and sometimes in the infinite time interval $t \geq t_0$.

– If there are inequalities that must be verified (this is the case with problems with unilateral joints or friction), then it often happens that there is no solution or that the solution exists but is not unique.

6.5. Equations of motion

In practice, as the unknown constraint efforts appear as linear variables in Lagrange's equations, one can always manage to eliminate these efforts in favor of the kinematic unknowns q. It is assumed here that this operation is carried out and we thus arrive at a system of n equations for n unknowns q.

Definition. *An equation of motion* for the system $\mathcal{S}$ is an equation that contains only $(q, \dot{q}, \ddot{q}, t)$ and no constraint efforts. This is a differential equation in $q(t)$.

The equations of motion take the form

$$\Phi_i(t, q, \dot{q}, \ddot{q}) = 0,\ i \in [1, n]$$

Once the equations of motion are solved, one can calculate the constraint efforts as functions of $(q, \dot{q}, \ddot{q}, t)$.

6.6. Example 1

We work in a Galilean reference frame $R_g = R_0$ endowed with the coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and we consider a disk S of center C, of radius R, homogeneous and of mass m (Figure 6.1). The disk is moving in the plane $O\vec{x}_0\vec{y}_0$ and remains in contact with axis $O\vec{x}_0$ at point I. It is subjected to the gravity field $-g\vec{y}_0$ and a constant torque $\Gamma\vec{z}_0$. The contact efforts at I are $T\vec{x}_0 + N\vec{y}_0$ (N is the normal contact force, T is the tangential contact force) and $\Gamma_r\vec{z}_0$ (Γ_r is the torque due to the *rolling* resistance).

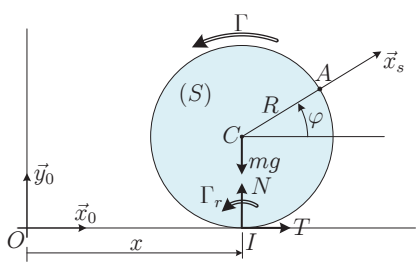


Figure 6.1. *Disk rolling along an axis*

The *a priori* position of the disk in R_0 is defined by the coordinates (x, y) of the center C and the angle of rotation φ of the disk, equal to the angle between $\vec{x}_0$ and a radius $\overrightarrow{CA}$ attached to the disk. The contact between the disk and the axis $O\vec{x}_0$ is expressed through the constraint equation $y = R$ (a holonomic relationship). The disk has, thus, 2 degrees of freedom.

We choose the following parameterization:

PARAMETERIZATION.

- The primitive parameters are (x, y, φ).
- We put the constraint equation $y = R$ in primitive category, which enables us to eliminate y in the calculations.
- The retained parameters for the problem are, therefore, $q = (x, \varphi)$.
- There is no complementary constraint equation.

The parameterized kinetic energy of the disk S with respect to R_0 is

$$E^c_{0S} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\varphi}^2 \underset{y=R}{=} \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\varphi}^2,$$

where $I \equiv \frac{1}{2}mR^2$ is the moment of inertia of S about axis $C\vec{z}_0$. The weight of the system is derivable from the potential

$$\mathcal{V}_0 = mgy + const \underset{y=R}{=} const,$$

where $const$ denotes an arbitrary constant. The VP of the given torque Γ is

$$\begin{aligned}\mathscr{P}^*(\Gamma\vec{z}_0) &= \Gamma\vec{z}_0.\vec{\Omega}^*_{0S} \quad \text{with } \vec{\Omega}^*_{0S} = \dot{\varphi}^*\vec{z}_0,\\ &= \Gamma\dot{\varphi}^*\end{aligned}$$

which gives $(D'_1, D'_3) = (0, \Gamma)$.

What now remains to be calculated is the VP of the constraint efforts (that is, the contact forces N, T and the torque Γ_r due to the rolling resistance):

$$\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to S}) = (T\vec{x}_0 + N\vec{y}_0).\vec{V}^*_{0S}(I) + \Gamma_r\vec{z}_0.\vec{\Omega}^*_{0S}$$

Note, in passing, that the reference frame R_0 appears in the indices of the VV $\vec{V}^*_{0S}(I)$ and virtual angular velocity $\vec{\Omega}^*_{0S}$. However, the right-hand side does not actually depend on R_0 as the hypothesis [2.33] is satisfied.

The VV at I is calculated using [4.35]: $\vec{V}^*_{0S}(I) = \vec{V}^*_{0S}(C) + \vec{\Omega}^*_{0S} \times \overrightarrow{CI}$, where the VV at the center C is given by [4.11]:

$$\begin{aligned}\vec{V}^*_{0S}(C) &= \frac{\partial\overrightarrow{OC}}{\partial x}\dot{x}^* + \frac{\partial\overrightarrow{OC}}{\partial\varphi}\dot{\varphi}^* \quad \text{with} \quad \overrightarrow{OC} = x\vec{x}_0 + y\vec{y}_0 = x\vec{x}_0 + R\vec{y}_0\\ &= \dot{x}^*\vec{x}_0\end{aligned}$$

Hence $\vec{V}^*_{0S}(I) = (\dot{x}^* + R\dot{\varphi}^*)\vec{x}_0$. Consequently, the VP of the constraint efforts is

$$\begin{aligned}\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to S}) &= (T\vec{x}_0 + N\vec{y}_0).\vec{V}^*_{0S}(I) + \Gamma_r\vec{z}_0.\vec{\Omega}^*_{0S}\\ &= T\dot{x}^* + (\Gamma_r + TR)\dot{\varphi}^*\end{aligned}$$

which gives $(L_1, L_2) = (T, \Gamma_r + TR)$.

Lagrange's equations [6.5] are, thus, written as

$$\begin{aligned} \mathscr{L}_x : \ m\ddot{x} &= T \\ \mathscr{L}_\varphi : \ I\ddot{\varphi} &= \Gamma + \Gamma_r + TR \end{aligned}$$

where the symbol $\mathscr{L}_x$, for instance, denotes Lagrange's equation corresponding to $q_i = x$.

We only have two Lagrange's equations for four unknowns: two kinematic unknowns x, φ and two unknown constraint efforts T, Γ_r, i.e. two equations are lacking for us to be able to solve the problem. The lacking equations are given by the contact laws at point I, for example:

– It may be assumed that the contact surface is rough enough at I for the slipping to be zero at any instant, which is expressed by the constraint equation $\dot{x} + R\dot{\varphi} = 0$ (non-holonomic equation). We can also assume that there is no friction, such that $T = 0$ at any instant.

– It may be assumed that the torque due to the rolling resistance is governed by the law $\Gamma_r = hN$, where h is the rolling resistance coefficient (the counterpart of the coefficient of friction), provided that N is known. Simply, it may be assumed that there is no rolling resistance, $h = 0$, such that $\Gamma_r = 0$ at any instant.

These contact hypotheses will be specified in the following example.

6.7. Example 2

We return to the previous example of the disk in contact with an axis, assuming that there is no slip at the contact point I at any instant and no rolling resistance, $\Gamma_r = 0$. The problem is represented by Figure 6.1, where the torque Γ_r is removed.

The problem has two constraint equations: the geometric contact condition at I, $y = R$ (holonomic relationship) and the no-slip condition $\dot{x} + R\dot{\varphi} = 0$ (semi-holonomic relationship). The holonomic relationship may be classified either as a primitive equation or a complementary equation. On the other hand, as the non-holonomic equation cannot be used to eliminate a position parameter, it must be classified as a complementary equation.

We will write the equations considering two different parameterizations – the first is a reduced parameterization and the second is a total one, as per definition [2.18] – and we will see the repercussions of the choice of parameterization on the equation obtained.

6.7.1. *Reduced parameterization*

We choose the following reduced parameterization, which is similar to the parameterization chosen in the previous example, except that, here, there is also the additional no-slip condition, which appears as a complementary constraint equation:

PARAMETERIZATION.

- The primitive parameters are (x, y, φ).
- The constraint equation $y = R$ is classified as a primitive equation.
- The retained parameters of the problem are, thus, $q = (x, \varphi)$.
- The complementary constraint equation is $\dot{x} + R\dot{\varphi} = 0$.

The parameterized kinetic energy and the potential have the same expressions as in the previous example:

$$E^c_{0S} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I\dot{\varphi}^2 \underset{y=R}{=} \frac{1}{2}m\dot{x}^2+\frac{1}{2}I\dot{\varphi}^2 \qquad \text{and} \qquad \mathcal{V}_0 = mgy + const \underset{y=R}{=} const$$

We can use the primitive constraint equation $y = R$ to eliminate y from the expressions. On the contrary, *at this stage*, we *cannot* use the complementary constraint equation $\dot{x} + R\dot{\varphi} = 0$ to eliminate, for example, $\dot{\varphi}$ in favor of $\dot{x}$ in the expression for E^c_{0S}. The complementary constraint equations can only be used *after* differentiating the kinetic energy.

The VP of the contact efforts at I is the same as in the previous example, except that here the torque Γ_r is zero:

$$\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to S}) = T\dot{x}^* + TR\dot{\varphi}^*$$

Lagrange's equations [6.5] thus give

$$\begin{aligned} \mathscr{L}_x :\ m\ddot{x} &= T \\ \mathscr{L}_\varphi :\ I\ddot{\varphi} &= \Gamma + TR \end{aligned}$$

Taking account of the complementary constraint equation $\dot{x} + R\dot{\varphi} = 0$, we have three second-order differential equations in time for two kinematic unknowns x, φ and the force unknown T. We easily obtain

$$\ddot{x} = -\frac{2\Gamma}{3mR} \qquad \ddot{\varphi} = \frac{2\Gamma}{3mR^2} \qquad T = -\frac{2\Gamma}{3R} \qquad [6.10]$$

6.7.2. *Total parameterization*

We will now change the parameterization by classifying the constraint equation $y = R$ as a complementary equation:

PARAMETERIZATION.

- The primitive parameters are still x, y, φ.
- There is no primitive constraint equation.
- Consequently, the retained parameters are the same: x, y, φ.
- The complementary constraint equations are $y = R$ and $\dot{x} + R\dot{\varphi} = 0$.

The parameterized kinetic energy and the potential are

$$E^c_{0S} = \frac{1}{2}m(\dot{x}^2+\dot{y}^2)+\frac{1}{2}I\dot{\varphi}^2 \qquad \text{and} \qquad \mathcal{V}_0 = mgy + const$$

Since we *are not* allowed to use the complementary constraint equations *at this stage*, we cannot transform these expressions as was done in the first parameterization. The expression of the VVF also changed with respect to that of the first parameterization. According to [4.11]:

$$\begin{aligned} \vec{V}^*_{0S}(C) &= \frac{\overrightarrow{\partial C}}{\partial x}\dot{x}^* + \frac{\overrightarrow{\partial C}}{\partial y}\dot{y}^* + \frac{\overrightarrow{\partial C}}{\partial \varphi}\dot{\varphi}^* \quad \text{with } \overrightarrow{OC} = x\vec{x}_0 + y\vec{y}_0 \\ &= \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0 \end{aligned}$$

Hence, using [4.35]:

$$\vec{V}^*_{0S}(I) = \vec{V}^*_{0S}(C) + \vec{\Omega}^*_{0S} \times \overrightarrow{CI} = (\dot{x}^* + R\dot{\varphi}^*)\vec{x}_0 + \dot{y}^*\vec{y}_0$$

The VP of the constraint efforts at I is thus

$$\begin{aligned}\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to S}) &= (T\vec{x}_0 + N\vec{y}_0).\vec{V}^*_{0S}(I) + \Gamma_r\vec{z}_0.\vec{\Omega}^*_{0S} \\ &= T\dot{x}^* + N\dot{y}^* + (\Gamma_r + TR)\dot{\varphi}^*\end{aligned}$$

Lagrange's equations [6.5] write

$$\begin{aligned}\mathscr{L}_x :\ m\ddot{x} &= T \\ \mathscr{L}_y :\ m\ddot{y} &= N - mg \\ \mathscr{L}_\varphi :\ I\ddot{\varphi} &= \Gamma + TR\end{aligned}$$

to which are added the complementary constraint equations $y = R$ and $\dot{x} + R\dot{\varphi} = 0$. We thus have five equations for three kinematic unknowns x, φ and two force unknowns N, T. By solving these equations, we arrive again at the same results [6.10] for x, φ, T as in the first parameterization and, in addition, we find the normal contact force N:

$$N = mg$$

6.7.3. *Comparing the two parameterizations*

The two parameterizations we have studied differ only in how the constraint equation $y = R$ is classified: in the first parameterization, this equation is treated as a primitive constraint equation, while in the second it is considered as complementary constraint equation. It can be seen that the information obtained varies in richness, depending on which parameter is chosen:

1. With the first parameterization, which is a reduced parameterization, we have three equations for three unknowns and we are able to determine the position parameters x, φ and the tangential contact force T.

2. With the second, which is a total parameterization, we have five equations for five unknowns. We are able to obtain the same quantities as before and, in addition, we obtain the expression for the normal contact force N, which cannot be obtained with the first parameterization.

If we do not wish to study the force N, it is sufficient to choose the reduced parameterization. If we do wish to study this force, then we must choose the second parameterization in which the constraint equation $y = R$, ensured by the force N itself, is classified as a complementary equation.

Figuratively speaking, putting the constraint equation $y = R$ into complementary equations amounts to "*virtually releasing the contact constraint at I*" so as to "*make the contact force N do virtual power*".

In analytical mechanics, parameterization, as defined in [2.19]*, is the first task to be dealt with in solving a problem and is a fundamental task that is incumbent on the physicist to carry out.* In general, the primitive parameters are quite obvious in view of the geometry of the problem and the constraint equations can easily be identified in view of the existing mechanical joints. What then remains to be chosen is the classification of the constraint equations: putting this equation in the primitive category and another in the complementary category. The physicist is free to choose the parameterization, i.e. the classification of the constraint equations. They must simply choose an appropriate parameterization that will provide the desired information.

As has just been seen in this example, if we wish to obtain the normal contact force – in order to know, for example, whether the contact at I is persistent – we must choose the second parameterization and not the first.

6.8. Example 3

In the following example, the calculation of the constraint forces is a little more complicated than in the previous example.

Consider a system $\mathcal{S}$ made up of two rigid bodies, a cart (C) and a particle p, as shown in Figure 6.2. The system is in plane motion in a Galilean reference frame $R_g = R_0$ endowed with the coordinate system $(O; \vec{x}_g, \vec{y}_g, \vec{z}_g)$.

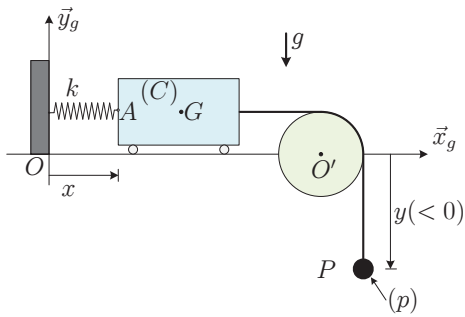


Figure 6.2. *Cart connected to a particle*

- The system $\mathcal{S}$ is subjected to the gravity field $-g\vec{y}_g$.

- The mass of the cart (C) is M and its center of mass is G. It is assumed that the connection between the cart and its wheels are such that the constraint efforts exerted by the wheels on the cart are reduced to vertical forces.

- The particle p has mass m.

- The cart is attached to a massless spring of stiffness k. It is assumed that the spring is unstretched when $x = 0$.

  One end A of the spring is attached to (C) and the other end to the axis $O\vec{y}_g$ fixed in R_g.

- (C) and p are connected to each other by a flexible, inextensible wire with no mass. This wire rests on a pulley which has no mass and which rotates, without friction, around the $O'\vec{z}_g$ axis.

The *a priori* position of the system in R_g is defined by the abscissa x of point A, which represents the position of the cart, and the ordinate y of the particle. Since the ordinate y is measured parallel to vector $\vec{y}_g$, it is negative in Figure 6.2.

The constraint equation that expresses the rope connection between (C) and p is written as follows (note that $y < 0$ and thus $|y| = -y$):

$$x + y = const \quad \leftrightarrow \quad y = const - x,$$

where $const$ denotes a known, fixed constant that does not need to be specified.

As in the previous example, we will write the equations considering two different parameterizations and we will compare the respective results.

6.8.1. *Independent parameterization*

Let us start by choosing the following independent parameterization:

PARAMETERIZATION.

- The primitive parameters are x, y.
- The constraint equation $x + y = const$ due to the rope is classified as primitive. This then allows us to express y as a function of x through $y = const - x$.
- The retained parameter of the problem is, therefore, x.
- There is *no* complementary constraint equation.

The parameterized kinetic energy $E^c_{g\mathcal{S}}(x, \dot{x})$ is

$$\begin{aligned} E^c_{g\mathcal{S}} &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \quad \text{with } \dot{y} = -\dot{x} \\ &= \frac{1}{2}(M+m)\dot{x}^2 \end{aligned} \qquad [6.11]$$

The weight of the system is derivable from the potential

$$\begin{aligned} \mathcal{V}_g(\mathcal{F}_{weight\to\mathcal{S}}) &= Mgy_G + mgy + const \quad \text{where} \begin{cases} y_G, \text{ the ordinate of } G, \text{ is constant} \\ y = const - x \end{cases} \\ &= -mgx + const \end{aligned}$$

Further, according to [5.27] and taking into account the fact that the spring is unstretched when $x = 0$, the restoring force on the cart due to the spring is derivable from the potential $\mathcal{V}_g(\mathcal{F}_{spring\to C}) = \frac{1}{2}kx^2 + const$. The total potential is, therefore:

$$\mathcal{V}_g = -mgx + \frac{1}{2}kx^2 + const \qquad [6.12]$$

What remains is to calculate the VP of the efforts that are not derivable from a potential. These are (i) the forces exerted by the wheels on the cart and (ii) the tensions in the rope exerted on the cart and on the particle (recall that the spring, the rope and the wheels of the cart are not part of the mechanical system being studied).

– The forces exerted on the cart by the wheels are vertical with magnitudes Y_1, Y_2 (see Figure 6.3(a)).

– As concerns the tension in the rope, it has the same value T on the cart as well as on the particle, as the rope and the pulley have no mass and the pulley rotates about its axis without friction.

  REMARK. This assertion can be proved either by writing Lagrange's equation for the pulley alone or by writing the moment equation (from Newton's laws) about axis $O'\vec{z}_g$, for the pulley alone. Let us describe how this is done using Newton's laws. Let T_1, T_2 denote the tensions in the rope exerted, respectively, on the cart and on the particle (Figure 6.3(b)). As the pivot between the pulley and its axis is frictionless, the resultant force of the constraint efforts exerted by the support on the pulley is $\vec{R}_{O'} \equiv \vec{R}_{support\to pulley}$ passing through O' and the resultant moment about O' is zero. By applying Newton's laws on the pulley, for instance, we have, using the obvious notations, $I\ddot{\varphi} = a(T_1 - T_2)$. As the pulley has no mass, the moment of inertia I is zero and we obtain $T_1 = T_2$, denoted by T. ■

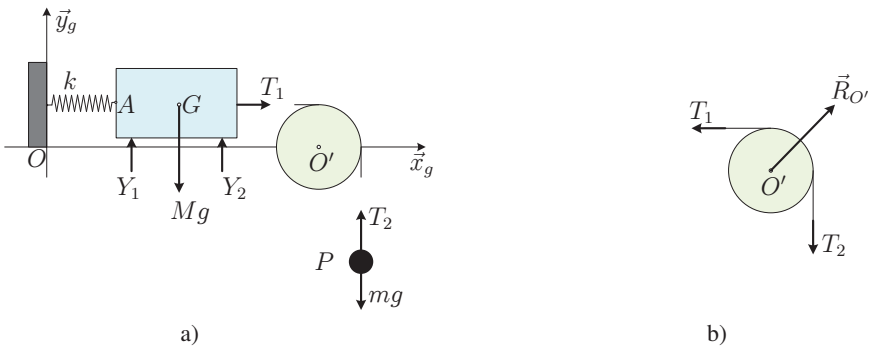


Figure 6.3. *External efforts on the cart and particle*

The VP is calculated in the VVF [4.11] associated with the chosen parameterization:

– VVF of the cart (C): $\forall$ particle p' of the cart, of position P', $\vec{V}^*(p') = \dfrac{\overrightarrow{\partial P'}}{\partial x}\dot{x}^* = \dot{x}^*\vec{x}_g$;

– VVF of the particle p:

$$\begin{aligned}\vec{V}^*(p) &= \frac{\overrightarrow{\partial P}}{\partial x}\dot{x}^*, \quad \text{where } \overrightarrow{OP} = const\vec{x}_g + y\vec{y}_g = const\vec{x}_g + (const - x)\vec{y}_g \\ &= -\dot{x}^*\vec{y}_g\end{aligned}$$

From this, we can derive the VP of efforts non-derivable from a potential:

$$\left.\begin{array}{l}\mathscr{P}^*(\mathcal{F}_{wheels\to C}) = (Y_1\vec{x}_g).(\dot{x}^*\vec{x}_g) + (Y_2\vec{x}_g).(\dot{x}^*\vec{x}_g) = 0 \\ \left.\begin{array}{l}\mathscr{P}^*(\mathcal{F}_{rope\to C}) = (T\vec{x}_g).(\dot{x}^*\vec{x}_g) \\ \mathscr{P}^*(\mathcal{F}_{rope\to p}) = (T\vec{y}_g).(-\dot{x}^*\vec{y}_g)\end{array}\right\} \Rightarrow \mathscr{P}^*(\mathcal{F}_{rope\to \mathcal{S}}) = 0\end{array}\right\} \Rightarrow \mathscr{P}^*(\mathcal{F}_{\text{wheels and rope}}) = 0 \qquad [6.13]$$

REMARK. Anticipating Chapter 7, we know that the connection by a flexible, inextensible massless wire is a *perfect* joint (see definition [7.79]) and that the considered VVF is compatible with this joint as there is no complementary constraint equation. From this, it can be directly concluded that $\mathscr{P}^*(\mathcal{F}_{rope\to \mathcal{S}\,=\,C\,\cup\,p}) = 0$. ■

Taking into account [6.11]–[6.13], Lagrange's equation [6.5] corresponding to x can be written as

$$(M + m)\ddot{x} + kx = mg \qquad [6.14]$$

This is a second-order time differential equation for the kinematic unknown x. By denoting $\dfrac{k}{M+m} = \omega_0^2$ and $\dfrac{mg}{M+m} = \omega_0^2 d$ ($\omega_0 > 0$ and $d > 0$), the previous equation can be recast as

$$\ddot{x} + \omega_0^2 x = \omega_0^2 d$$

By taking into account the initial conditions $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$, where x_0 and $\dot{x}_0$ are given, we obtain

$$x(t) = (x_0 - d)\cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t + d \qquad [6.15]$$

The cart and the particle are in oscillatory motion, the circular frequency ω_0 of which depends on the stiffness of the spring and on the total mass of the system.

6.8.2. *Total parameterization*

We will now work with another parameterization, where the constraint equation $x + y = const$ due to the rope is, this time, classified as a complementary equation:

PARAMETERIZATION.

- The primitive parameters are still x and y.
- There is no primitive constraint equation.
- Consequently, the retained parameters are the same: x and y.
- The complementary constraint equation is $x + y = const$.

This time, the parameterized kinetic energy is, *a priori*, a function of $x, y, \dot{x}, \dot{y}$:

$$E^c_{g\mathcal{S}}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{y}^2 \qquad [6.16]$$

The potential of the spring is the same as in the first parameterization, while the potential due to the weight of the system does change:

$$\begin{aligned} \mathcal{V}_g(\mathcal{F}_{weight\to\mathcal{S}}) &= Mgy_G + mgy + const \quad \text{where } y_G \text{ is constant} \\ &= mgy + const \end{aligned}$$

The total potential is, therefore:

$$\mathcal{V}_g = mgy + \frac{1}{2}kx^2 + const \qquad [6.17]$$

The VVF associated with the parameterization also changes:

– The VVF of the cart (C): $\forall$ particle p' of the cart whose position is P',

$$\vec{V}^*(p') = \underbrace{\frac{\overrightarrow{\partial P'}}{\partial x}}_{\vec{x}_g}\dot{x}^* + \underbrace{\frac{\overrightarrow{\partial P'}}{\partial y}}_{\vec{0}}\dot{y}^* = \dot{x}^*\vec{x}_g$$

– VVF of the particle p:

$$\vec{V}^*(P) = \underbrace{\frac{\overrightarrow{\partial P}}{\partial x}}_{\vec{0}}\dot{x}^* + \underbrace{\frac{\overrightarrow{\partial P}}{\partial y}}_{\vec{y}_g}\dot{y}^* = \dot{y}^*\vec{y}_g$$

Consequently, the VP in this VVF of the efforts that are not derivable from a potential is

$$\left.\begin{array}{l} \mathscr{P}^*(\mathcal{F}_{wheels\to C}) = 0 \\ \mathscr{P}^*(\mathcal{F}_{rope\to C}) = (T\vec{x}_g).(\dot{x}^*\vec{x}_g) = T\dot{x}^* \\ \mathscr{P}^*(\mathcal{F}_{rope\to p}) = (T\vec{y}_g).(\dot{y}^*\vec{y}_g) = T\dot{y}^* \end{array}\right\} \Rightarrow \mathscr{P}^*(\mathcal{F}_{\text{wheels and rope}}) = (T - kx)\dot{x}^* + (T - mg)\dot{y}^* \qquad [6.18]$$

By taking into account [6.16]–[6.18], Lagrange's equations [6.5] are written as

$$\begin{array}{l} \mathscr{L}_x : \ M\ddot{x} = T - kx \\ \mathscr{L}_y : \ m\ddot{y} = T - mg \end{array}$$

These equations, combined with the complementary constraint equation $x + y = const$ (which can only be used *now*), make up three equations for three unknowns: the two kinematic unknowns x, y and the unknown constraint effort, which is the tension T in the rope. By solving these equations, we once again arrive at the equation of motion [6.14] and we also obtain the relationship for the tension T:

$$T = mg + m\omega_0^2(x - d)$$

Hence, taking into account [6.15]:

$$T = mg + m\omega_0^2 \left[(x_0 - d)\cos\omega_0 t + \frac{\dot{x}_0}{\omega_0}\sin\omega_0 t\right] = mg + m\omega_0^2 C\cos(\omega_0 t - \varphi)$$

with $C \equiv \sqrt{(x_0 - d)^2 + \frac{\dot{x}_0^2}{\omega_0^2}}$. The expression for T allows us to find the inequality that the initial conditions $x_0, \dot{x}_0$ must satisfy in order for the rope to remain taut:

$$\forall t \quad T \geq 0 \quad \Leftrightarrow \quad mg \geq m\omega_0^2 C \quad \Leftrightarrow \quad (x_0 - d)^2 + \frac{\dot{x}_0^2}{\omega_0^2} \leq \left(1 + \frac{M}{m}\right)^2 d^2$$

6.8.3. *Comparing the two parameterizations*

As in the previous example, it is seen that given the same mechanical system, the richness of the information obtained depends on the chosen parameterization. Thus, *the parameterization is a preliminary task, which is of primordial importance in analytical mechanics.* Here, if we wish to study only the motion of the system, it is sufficient to choose the independent parameterization. On the other hand, if we wish to also find the tension T in the rope, we must choose the second parameterization in which the constraint equation $x + y = const$, which is ensured by the very tension T, is classified as a complementary equation.

Figuratively speaking, classifying the constraint equation $x + y = const$ as a complementary equation amounts to "*virtually cutting the rope*", so as to *make the tension T do virtual power*.

REMARK. Anticipating Chapter 7, we know that the connection by a rope, considered in this example, is a perfect joint. On the other hand, we also know that the VVF associated with the second parameterization is not compatible with this joint. This leads to a non-zero VP for the tension T and to the presence of T in Lagrange's equations. ■

6.9. Working in a non-Galilean reference frame

In certain cases, we may have to work in a non-Galilean (or non-inertial) reference frame R_1. The Lagrange's equations relative to such a reference frame are given by the following theorem.

Theorem. Let $\mathcal{S}$ be a system made up of one or more rigid bodies and *endowed with the parameterization* [2.19]. Consider two reference frames, R_g, which is Galilean, and R_1, which is non-Galilean.

HYPOTHESES:

(i) We choose $R_0 = R_g$ according to convention [6.1].

(ii) It is assumed that the non-Galilean reference frame R_1 verifies [2.26], i.e. the rotation tensor $\bar{\bar{Q}}_{01}$ does *not* depend on q.

Then, the Lagrange's equations for the system $\mathcal{S}$ are written in the non-Galilean reference frame R_1 as follows:

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{d}{dt}\frac{\partial E^c_{R_1\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_1\mathcal{S}}}{\partial q_i} = Q_i - C_i^{(R_gR_1)} - C_i^{(Coriolis)}} \qquad [6.19]$$

where:

– $E^c_{R_1\mathcal{S}}(q, \dot{q}, t)$ is the parameterized kinetic energy of the system $\mathcal{S}$ with respect to R_1, defined by [2.54];

– Q_i is the ith generalized force of the efforts

$$\boxed{Q_i = \int_{\mathcal{S}} \vec{f}(P,t).\frac{\overrightarrow{\partial P}}{\partial q_i} dm + \sum_s \int_{S_s} \vec{c}(P,t) dm \,.\vec{\omega}^i_{1s}}$$

– $C_i^{(R_gR_1)}$ is, by definition, $\boxed{C_i^{(R_gR_1)} \equiv \int_{\mathcal{S}} \vec{\Gamma}_{R_gR_1}.\frac{\overrightarrow{\partial P}}{\partial q_i} dm}$, where $\vec{\Gamma}_{R_gR_1}(P,t)$ is the so-called *background acceleration*, as defined in [1.72];

– $C_i^{(Coriolis)}$ is, by definition, $\boxed{C_i^{(Coriolis)} \equiv \int_{\mathcal{S}} \left[2\vec{\Omega}_{R_gR_1} \times \vec{V}_{R_1\mathcal{S}}(P)\right] . \frac{\overrightarrow{\partial P}}{\partial q_i} dm}$, where $\vec{\Gamma}_{Coriolis} \equiv 2\vec{\Omega}_{R_gR_1} \times \vec{V}_{R_1\mathcal{S}}(P)$ is the Coriolis acceleration.

The Lagrange's equations can, thus, be written in a non-Galilean reference frame, provided that the generalized forces due to the background inertial forces and the Coriolis inertial forces are added to the right-hand side.

PROOF. The reasoning is similar to that used for the Lagrange's equations [6.2]. As R_g is Galilean, the PVP [5.1] may be applied in R_g:

$$\forall t, \forall \text{ CVV } V^*_{R_g\mathcal{S}}, \qquad \mathscr{P}^*(\rho\vec{\Gamma}_{R_g\mathcal{S}}) = \mathscr{P}^*(\mathcal{F}_{\to\mathcal{S}}) \qquad [6.20]$$

• According to [5.16], the VP of the effort system $\mathcal{F}_{\to\mathcal{S}}$ in the VVF $V^*_{R_g\mathcal{S}}$ has the form

$$\mathscr{P}^*(\mathcal{F}_{\to\mathcal{S}}) = \sum_{i=1}^{n} Q_i \dot{q}^*_i \qquad [6.21]$$

where, because hypothesis [2.26] (written with R_g instead of R_1) is satisfied because of hypothesis (i), the coefficient Q_i is given by [5.18]:

$$Q_i = \int_{\mathcal{S}} \vec{f}(P,t).\frac{\overrightarrow{\partial O_gP}}{\partial q_i} dm + \sum_s \int_{S_s} \vec{c}(P,t) dm \,.\vec{\omega}^i_{1s}$$

Furthermore, the point O_g fixed in R_g may be omitted as it is assumed independent of q.

• On the other hand, on account of the composition formula for accelerations [1.71]:

$$\vec{\Gamma}_{R_g\mathcal{S}}(P,t) = \vec{\Gamma}_{R_gR_1}(P,t) + \vec{\Gamma}_{R_1\mathcal{S}}(P,t) + \underbrace{2\vec{\Omega}_{R_gR_1} \times \vec{V}_{R_1\mathcal{S}}(P,t)}_{\equiv\vec{\Gamma}_{Coriolis}}$$

the VP of the quantities of acceleration $\mathscr{P}^*(\rho\vec{\Gamma}_{R_g\mathcal{S}})$, at an instant t, in the VVF $V^*_{R_g\mathcal{S}}$, takes the form

$$\begin{aligned}\mathscr{P}^*(\rho\vec{\Gamma}_{R_g\mathcal{S}}) &= \int_{\mathcal{S}} \vec{\Gamma}_{R_g\mathcal{S}} \cdot \vec{V}^*_{R_g\mathcal{S}} \mathrm{d}m \\ &= \underbrace{\int_{\mathcal{S}} \vec{\Gamma}_{R_gR_1} \cdot \vec{V}^*_{R_g\mathcal{S}} \mathrm{d}m}_{\boxed{1}} + \underbrace{\int_{\mathcal{S}} \vec{\Gamma}_{R_1\mathcal{S}} \cdot \vec{V}^*_{R_g\mathcal{S}} \mathrm{d}m}_{\boxed{2}} + \underbrace{\int_{\mathcal{S}} \left[2\vec{\Omega}_{R_gR_1} \times \vec{V}_{R_1\mathcal{S}}(P)\right] \cdot \vec{V}^*_{R_g\mathcal{S}} \mathrm{d}m}_{\boxed{3}}\end{aligned} \qquad [6.22]$$

where the terms $\boxed{1}$, $\boxed{2}$, $\boxed{3}$ will be developed in detail below.

For term $\boxed{2}$, note that [4.44] is applicable by virtue of the adopted hypotheses: the virtual velocity does not depend on the reference frame and can be written as

$$\vec{V}^*_{R_g\mathcal{S}}(P) = \vec{V}^*_{R_1\mathcal{S}}(P) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i} \dot{q}_i^* \qquad [6.23]$$

Hence

$$\boxed{2} \equiv \int_{\mathcal{S}} \vec{\Gamma}_{R_1\mathcal{S}} \cdot \vec{V}^*_{R_g\mathcal{S}} \mathrm{d}m = \int_{\mathcal{S}} \vec{\Gamma}_{R_1\mathcal{S}} \cdot \vec{V}^*_{R_1\mathcal{S}} \mathrm{d}m \equiv \mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}})$$

In the previous relationship, we wrote the reference frame index R_1 in the VP ($\mathscr{P}^*_{R_1}(\rho\vec{\Gamma}_{R_1\mathcal{S}})$) in order to make it easier to follow the reasoning. However, as the VP is, in fact, independent of the reference frame, the index R_1 will be removed. Hypothesis (ii), which is hypothesis [2.26], allows us to apply [5.38]:

$$\boxed{2} = \mathscr{P}^*(\rho\vec{\Gamma}_{R_1\mathcal{S}}) = \sum_{i=1}^{n} C_i \dot{q}_i^* \qquad \text{where} \qquad C_i \equiv \frac{d}{dt}\left(\frac{\partial E^c_{R_1\mathcal{S}}}{\partial \dot{q}_i}\right) - \frac{\partial E^c_{R_1\mathcal{S}}}{\partial q_i} \qquad [6.24]$$

As concerns terms $\boxed{1}$ and $\boxed{3}$, they are written taking into account [6.23]:

$$\begin{cases} \boxed{1} \equiv \displaystyle\int_{\mathcal{S}} \vec{\Gamma}_{R_gR_1} \cdot \vec{V}^*_{R_g\mathcal{S}} \mathrm{d}m = \sum_{i=1}^{n} C_i^{(R_gR_1)} \dot{q}_i^* \quad \text{or } C_i^{(R_gR_1)} \equiv \int_{\mathcal{S}} \vec{\Gamma}_{R_gR_1} \cdot \frac{\overrightarrow{\partial P}}{\partial q_i} dm \\ \boxed{3} \equiv \displaystyle\int_{\mathcal{S}} \left[2\vec{\Omega}_{R_gR_1} \times \vec{V}_{R_1\mathcal{S}}(P)\right] \cdot \vec{V}^*_{R_g\mathcal{S}} \mathrm{d}m = \sum_{i=1}^{n} C_i^{(Coriolis)} \dot{q}_i^* \\ \qquad \text{where } C_i^{(Coriolis)} \equiv \displaystyle\int_{\mathcal{S}} \vec{\Gamma}_{Coriolis} \cdot \frac{\overrightarrow{\partial P}}{\partial q_i} dm \end{cases} \qquad [6.25]$$

Inserting [6.24]–[6.25] into [6.22] yields

$$\mathscr{P}^*\left(\rho\vec{\Gamma}_{R_g\mathcal{S}}\right) = \sum_{i=1}^{n} \left[C_i^{(R_gR_1)} + C_i + C_i^{(Coriolis)}\right] q_i^* \qquad [6.26]$$

• By combining [6.21] and [6.26], the PVP [6.20] gives

$$\forall i \in [1, n], \quad C_i^{(R_gR_1)} + C_i + C_i^{(Coriolis)} = Q_i \qquad \blacksquare$$

A priori, the VP done by the Coriolis forces ($= -\boxed{3}$) is *not* zero, unlike the real power of the same forces, which is indeed zero:

$$-\int_{\mathcal{S}} \left[2\vec{\Omega}_{R_gR_1} \times \vec{V}_{R_1\mathcal{S}}(P)\right] \cdot \vec{V}_{R_1\mathcal{S}}(P) \mathrm{d}m = 0$$

7

Perfect Joints

In this chapter:

– we will define the concept of a VVF compatible with a mechanical joint,

– we will prove the invariance of the VVF compatible with a joint in different parameterizations,

– we can then define the concept of a perfect joint by using the previous results.

The results obtained in this chapter will be used to establish the Lagrange's equations in the presence of a perfect mechanical joint in Chapter 8.

For generality, in this chapter we will consider the VVs and the VPs with respect to any reference frame R_1, which is not necessarily Galilean. This enables us to establish the theory of perfect joints in a setting that is larger than that of Galilean reference frames. In the following chapters, where we will work in a Galilean reference frame R_g, one only has to apply the results obtained in this chapter by making $R_1 = R_g$.

• The common reference frame R_0 being chosen beforehand, it is assumed that the pair of reference frames (R_1, R_0) satisfies hypothesis [2.33]:

HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and the point O_1 fixed in R_1 does *not* depend on q.

As was recalled at the beginning of Chapter 6, hypothesis [2.33] implies that the VV of a particle and the VP of an effort system are *independent of the reference frame* R_1 and may thus be written *without the reference frame index*:

$$\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i} \dot{q}_i^* \qquad \text{and} \qquad \mathscr{P}^*(\mathcal{F}_{\to \mathcal{S}}, t)$$

In particular, the VP of the constraint efforts exerted by the mechanical joints on the system $\mathcal{S}$ is *independent of the reference frame* R_1 with respect to which it is calculated and is denoted by $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}, t)$. Hypothesis [2.33] is necessary as it implies the independence of the VP of efforts with respect to the reference and, as will be seen in section 7.5.1, this independence makes it possible to render the concept of a perfect joint intrinsic.

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

7.1. VFs compatible with a mechanical joint

7.1.1. *Definition*

Let $\mathcal{S}$ be a system composed of several rigid bodies, constrained by a certain number of mechanical joints, which may be joints between the rigid bodies in the system or joints between one rigid body in the system and another rigid body outside the system. We consider a given parameterization whose retained parameters are $q \equiv (q_1, \ldots, q_n)$ and t.

The existing mechanical joints are expressed by a certain number of constraint equations, which may be classified as primitive or complementary. It is assumed that the *complementary* constraint equations are either:

(i) *resolved*, holonomic constraint equations, for instance $q_n = \chi_n(q_1, \ldots, q_{n-1}, t)$,

(ii) unresolved, holonomic constraint equations of the form $f(q, t) = 0$,

(iii) or non-holonomic equations of the differential form $\sum_{i=1}^{n} \alpha_i(q, t)\, \dot{q}_i + \beta(q, t) = 0$.

In order to simplify the discussion, we decide to derive any constraint equation of the type (ii) with respect to time:

$$\sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t} = 0$$

such that it becomes a constraint equation of the type (iii), with $\alpha_i = \dfrac{\partial f}{\partial q_i}$, $i \in [1, n]$, and $\beta = \dfrac{\partial f}{\partial t}$. Thus, throughout the sequel, we will consider only two types of complementary constraint equations:

1. *resolved*, holonomic constraint equations, for instance $q_n = \chi_n(q_1, \ldots, q_{n-1}, t)$,
2. and non-holonomic constraint equations of the differential form $\sum_{i=1}^{n} \alpha_i(q, t)\, \dot{q}_i + \beta(q, t) = 0$.

[7.1]

The VV of the current particle p of the system associated with the previous parameterization is given by [4.18]: $\vec{V}^*(p) = \sum_{i=1}^{n} \dfrac{\overrightarrow{\partial P}}{\partial q_i} \dot{q}_i^*$. A new concept will now be defined, namely the VV associated with the parameterization and that is compatible with a mechanical joint.

Definition. [7.2]

Consider a given mechanical joint on the system. This joint is expressed by a certain number of constraint equations, which may be classified as primitive or complementary. However, only *complementary* equations are involved in this definition.

A VVF $\vec{V}^*(p)$ associated with the considered parameterization and *compatible with the considered joint* is defined as

$$\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i} \dot{q}_i^* \qquad [7.3]$$

where

1. the $(\dot{q}_i^*)_{1 \leq i \leq n}$ must satisfy the following relationships, all of which come from *complementary* constraint equations related to the *mechanical joint under consideration*:

(a) for any *resolved*, holonomic constraint equation, for instance $q_n = \chi_n(q_1, \ldots, q_{n-1}, t)$, the $(\dot{q}_i^*)_{1 \leq i \leq n}$, satisfies $\boxed{\dot{q}_n^* = \sum_{i=1}^{n-1} \frac{\partial \chi_n}{\partial q_i} \dot{q}_i^*}$, [7.4]

(b) for any non-holonomic, complementary constraint equation in the differential form $\sum_{i=1}^{n} \alpha_i \, \dot{q}_i + \beta = 0$, the $(\dot{q}_i^*)_{1 \leq i \leq n}$, satisfies $\boxed{\sum_{i=1}^{n} \alpha_i \dot{q}_i^* = 0}$. [7.5]

If there is a complementary constraint equation of the form $f(q, t) = 0$, the earlier relationship gives $\boxed{\sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \dot{q}_i^* = 0}$,

2. the retained parameters q, which appear in the $\frac{\partial \overrightarrow{P}}{\partial q_i}(q, t)$ of [7.3], the $\frac{\partial \chi_n}{\partial q_i}(q, t)$ of [7.4] (expressions obtained *after* differentiation), as well as in the $\alpha_i(q, t)$ of [7.5], must satisfy all existing *complementary* constraint equations related to the joint; namely, with the typical examples [7.1]:

$$q_n = \chi_n(q_1, \ldots, q_{n-1}, t) \qquad \text{and} \qquad \sum_{i=1}^{n} \alpha_i(q, t) \, \dot{q}_i + \beta(q, t) = 0 \qquad [7.6]$$

It is important to note that the previous definition brings into play

– only the constraint equations that are related to *the considered joint* and not those related to any other joints that may exist in the system,

– and only *complementary* constraint equations and not primitive equations.

Condition [7.4] is consistent with condition [7.5] since when $q_n = \chi_n(q_1, \ldots, q_{n-1}, t)$ is derived with respect to time, we obtain $\dot{q}_n = \sum_{i=1}^{n-1} \frac{\partial \chi_n}{\partial q_i} \dot{q}_i + \frac{\partial \chi_n}{\partial t}$ which has the form $\sum_{i=1}^{n} \alpha_i \dot{q}_i + \beta = 0$.

Consider a holonomic, complementary constraint equation $f(q, t) = 0$ and its differential form obtained through time derivation: $\dot{f} = \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial t} = 0$. The previous definition shows that the compatibility condition for a VVF resulting from equation $f = 0$ is, by definition, the condition resulting from the differential form $\dot{f} = 0$. Thus, if there is a semi-holonomic complementary constraint equation $\dot{f} = 0$, whose integrated form is $f = const$ (where $const$ is a constant of integration), then regardless of whether this constraint equation is used in the semi-holonomic form or in the integrated form, it engenders the same compatibility condition for a VVF.

According to definition [7.2], if there is no complementary equation for the mechanical joint (especially if the parameterization is independent) then conditions [7.4]–[7.6] are irrelevant and any VVF associated with the parameterization is automatically compatible with the joint.

Condition [7.6] is irrelevant if the $\frac{\partial \overrightarrow{P}}{\partial q_i}$ in [7.3], the $\frac{\partial \chi_n}{\partial q_i}$ in [7.4] and the α_i in [7.5] do not depend on q.

A VVF compatible with a joint is a specific VVF, subjected to the restrictive conditions [7.4]–[7.6]. The definition of a perfect joint will be based on the VVF compatible with the joint and not on the general VVF: the joint is said to be perfect if the VP of the constraint efforts is zero in any compatible VVF.

REMARK. Condition [7.6] will largely be used in this chapter and will also be used in Chapter 8, when we prove expression [8.5] for the generalized constraint forces L_i in the case of perfect joints.

Let us bring together two different rules we have encountered so far:

1. The rule stated in section 6.2: when we differentiate the kinetic energy and the potential, we can only use the complementary constraint equations *after* having obtained the derivatives.

2. Condition [7.6] seen above: we build up a VVF compatible with a joint by taking into account the complementary constraint equations relative to this joint.

These two rules, relating to two independent operations, are *not* contradictory.

Finally, the second rule has one point in common with the first: the complementary constraint equations are used *after* after obtaining the derivatives $\frac{\partial \overrightarrow{P}}{\partial q_i}(q,t)$ and $\frac{\partial \chi_n}{\partial q_i}(q,t)$. ■

According to definition [7.2], the VVF compatible with a given mechanical joint depends on the joint in question; it also depends on the chosen parameterization via the complementary constraint equations involved in the definition. In actual fact, as will be seen later in theorem [7.78], the VVF does *not* depend on the chosen parameterization. Consequently, the compatible VVF is intrinsic, in the sense that it depends only on the mechanical joint, and we can then simply speak of a VVF compatible with a joint.

7.1.2. *Generalizing the definition of a compatible VVF*

• Relationship [7.3] is not the only way of calculating the VV $\vec{V}^*(p)$ of a particle p. Depending on the situation, we may have to calculate $\vec{V}^*(p)$ using formula [4.32] for a rigid body or the composition formula for velocities [4.43]. Whatever the way chosen, the VV always takes the form of a linear combination of $\dot{q}_i^*$:

$$\vec{V}^*(p) = \sum_{i=1}^{n} (i\text{th vector, function of } (q,t))\ \dot{q}_i^* \qquad [7.7]$$

Definition [7.2] is modified in the following way to adapt to the VV of the previous form ($[\cdots]$ denote the sections that remain unchanged with respect to definition [7.2]):

Definition. A VVF $\vec{V}^*(p)$ associated with the considered parameterization and of the form [7.7] is *compatible with the considered joint* if

1. $[\cdots]$,

2. the retained parameters q, which appear in the ith vector of [7.7], the $\frac{\partial \chi_n}{\partial q_i}(q,t)$ of [7.4] (expressions obtained *after* differentiation), as well as the $\alpha_i(q,t)$ of [7.5], must satisfy all existing *complementary* constraint equations related to the joint $[\cdots]$.

• We may also be led to work in a reference frame R_1, which does not satisfy hypothesis [2.33]. In this case, the VV $\vec{V}^*_{R_1}(p)$ is not given by [7.3] but instead by the general definition [4.10]:

$$\vec{V}^*_{R_1}(p) \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)\dot{q}_i^*, \qquad [7.8]$$

where O_1 is a point fixed in R_1, the position vector $\overrightarrow{O_1P}$ is given by $\overrightarrow{O_1P} = \overrightarrow{O_1P}(q,t)$ and the rotation tensor $\bar{\bar{Q}}_{01}$ is a function of (q,t). Note, in passing, that this expression is also of the form [7.7].

The example of such a case can be seen in section 7.5.4 where we study a particle p moving along a hoop (C). The reference frame R_C defined by (C) is such that the $\bar{\bar{Q}}_{0C}$ depends on q and the VV $\vec{V}^*_{R_C}(p)$ of p with respect to R_C is given by [7.86].

As expression [7.8] does has not the form [7.3], we must specify what a VV $\vec{V}^*_{R_1}(p)$ compatible with a joint means. Definition [7.2] is generalized to the case of VVs [7.8] as follows:

Definition. [7.9]

A VVF $\vec{V}^*_{R_1}(p)$ associated with the considered parameterization and *compatible with the considered joint* is, by definition, the VVF [7.8], where

1. $[\cdots]$,
2. the retained parameters q, which appear in $\bar{\bar{Q}}_{01}.\dfrac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1P}\right)(q,t)$ of [7.8], the $\dfrac{\partial \chi_n}{\partial q_i}(q,t)$ of [7.4] (expressions obtained *after* differentiation) as well as in the $\alpha_i(q,t)$ of [7.5], must satisfy all existing *complementary* constraint equations related to the joint $[\cdots]$.

7.1.3. *Example 1 for VVFs compatible with a mechanical joint*

We work in the reference frame $R_1 = R_0$, endowed with an orthonormal coordinate system $(O;\vec{x}_0,\vec{y}_0)$ and consider a particle p which is constrained to move along a curve in the plane $O\vec{x}_0\vec{y}_0$, with the equation $y = \chi(x)$ (Figure 7.1). The position of the particle in the reference frame R_0 at a current instant t is denoted by P. The mechanical joint imposed on the particle here is represented by the single constraint equation $y = \chi(x)$.

Let us calculate the VVs of the particle that are compatible with the mechanical joint and associated with the following parameterization, where the constraint equation $y = \chi(x)$ is classified as a complementary equation:

TOTAL PARAMETERIZATION. [7.10]

- Primitive parameters: the Cartesian coordinates x, y of point P in the coordinate system $(O;\vec{x}_0,\vec{y}_0)$.
- Primitive constraint equations: none.
- Retained parameters: the same as the primitive parameters, that is: x, y. Hence, $P = P(x,y) = O + x\vec{x}_0 + y\vec{y}_0$.
- Complementary constraint equation: $y = \chi(x)$.

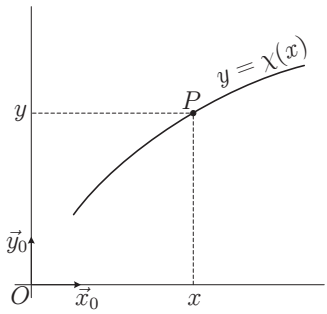


Figure 7.1. *Particle moving along a planar curve: parameterization using Cartesian coordinates*

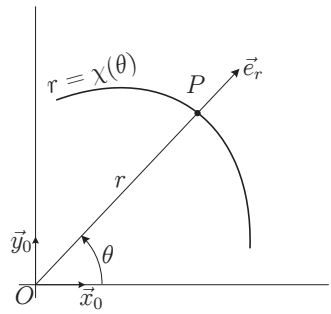


Figure 7.2. *Particle moving along a planar curve: parameterization using polar coordinates*

According to definition [7.2], a VV associated with this parameterization and compatible with the mechanical joint is

$$\vec{V}^*(p) = \frac{\overrightarrow{\partial P}}{\partial x}\dot{x}^* + \frac{\overrightarrow{\partial P}}{\partial y}\dot{y}^* = \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0 \qquad \text{where } \dot{y}^* = \frac{d\chi}{dx}(x)\,\dot{x}^*$$

As the vectors $\frac{\overrightarrow{\partial P}}{\partial x}$ and $\frac{\overrightarrow{\partial P}}{\partial y}$ do not depend on the retained parameters, condition [7.6] is irrelevant. Hence

$$\vec{V}^*(p) = \left[\vec{x}_0 + \frac{d\chi}{dx}(x)\vec{y}_0\right]\dot{x}^* \qquad [7.11]$$

7.1.4. *Example 2*

Let us go back to the previous example, but whilst working with polar coordinates this time. The curve along which the particle p is moving has the equation $r = \chi(\theta)$, where r, θ denote the polar coordinates of point P (Figure 7.2). Here, the mechanical joint imposed on the particle is expressed through the single constraint equation $r = \chi(\theta)$.

Let us calculate the VVs of the particle compatible with the mechanical joint and associated with the following parameterization, where the constraint equation $r = \chi(\theta)$ is classified as a complementary equation:

TOTAL PARAMETERIZATION. [7.12]

- Primitive parameters: the polar coordinates r, θ.
- Primitive constraint equations: none.
- Retained parameters: the same as the primitive parameters, that is: r, θ. Hence $P = P(\theta, r) = O + r\vec{e}_r(\theta)$.
- Complementary constraint equation: $r = \chi(\theta)$.

A VV associated with this parameterization and compatible with the mechanical joint is

$$\vec{V}^*(p) = \frac{\overrightarrow{\partial P}}{\partial r}\dot{r}^* + \frac{\overrightarrow{\partial P}}{\partial \theta}\dot{\theta}^* = \vec{e}_r(\theta)\dot{r}^* + r\vec{e}_\theta(\theta)\dot{\theta}^* \qquad \text{where} \begin{cases} \dot{r}^* = \dfrac{d\chi}{d\theta}(\theta)\,\dot{\theta}^* & \text{according to [7.4]} \\ \underline{\text{and } r = \chi(\theta)} & \text{according to [7.6]} \end{cases}$$

Definition [7.2] imposes condition [7.6], which consists of replacing the argument r appearing in $\frac{\overrightarrow{\partial P}}{\partial \theta} = r\vec{e}_\theta(\theta)$ with $r = \chi(\theta)$. Hence

$$\vec{V}^*(p) = \left[\frac{d\chi}{d\theta}(\theta)\,\vec{e}_r(\theta) + \chi(\theta)\,\vec{e}_\theta(\theta)\right]\dot{\theta}^* \qquad [7.13]$$

We can clearly see the difference between a general VV and a VV compatible with the joint:

– a general VV is written as $\vec{V}^*(p) = \vec{e}_r(\theta)\dot{r}^* + r\vec{e}_\theta(\theta)\dot{\theta}^*$, and it depends on the parameters r, θ and the virtual parameters $\dot{r}^*, \dot{\theta}^*$,

– a VV compatible with the constraint is given by [7.13], and it now depends only on θ and $\dot{\theta}^*$.

Generally speaking, as the compatible VVF is a particular VVF subjected to conditions [7.4]–[7.6], the expression for the compatible VVF is more restricted than that for the general VVF in the sense that it depends on fewer variables.

7.1.5. *Example 3: particle moving along a hoop rotating around a fixed axis*

We work in the reference frame $R_1 = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and a system consisting of a particle p located at P in R_0 and of a hoop (C) of center O, radius a and whose diameter lies constantly on the axis $O\vec{z}_0$.

$\overrightarrow{OP} = r\,\vec{z}$ denotes the position vector of p, where r is the radial distance and $\vec{z}$ is a unit vector (Figure 7.3). We define

– the unit vector $\vec{n}$, orthogonal to $\vec{z}_0$ and $\vec{z}$, such that the basis $(\vec{z}_0, \vec{z}, \vec{n})$ is right handed (but not necessarily orthonormal),

– the angle $\theta \equiv (\vec{z}_0, \vec{z})$ measured with respect to $\vec{n}$,

– the angle $\psi \equiv (\vec{x}_0, \vec{n})$ measured with respect to $\vec{z}_0$,

– the unit vector $\vec{v} \equiv \vec{z} \times \vec{n}$.

The primitive parameters of p are r, ψ, θ.

The position of the hoop is defined by the angle denoted by $\alpha \equiv (\vec{x}_0, \vec{c})$, measured around $\vec{z}_0$, where $\vec{c}$ is a unit vector orthogonal to the hoop ($O\vec{c}$ is thus the axis of the hoop).

Driven by an engine, the hoop rotates around the axis $O\vec{z}_0$ at a constant rate $\omega > 0$. At $t = 0$, we take $\vec{c} = \vec{x}_0$, such that $\alpha = \omega t$.

The particle p is constrained to remain on the hoop (C). The mechanical joint imposed on the particle p is expressed through two constraint equations $r = a$ and $\psi = \alpha$ (i.e. $\vec{n} = \vec{c}$), which are *resolved* equations.

Let us calculate the VVs of the particle compatible with the mechanical joint and associated with the following parameterization:

REDUCED PARAMETERIZATION. [7.14]

- Primitive parameters: r, ψ, θ and α.

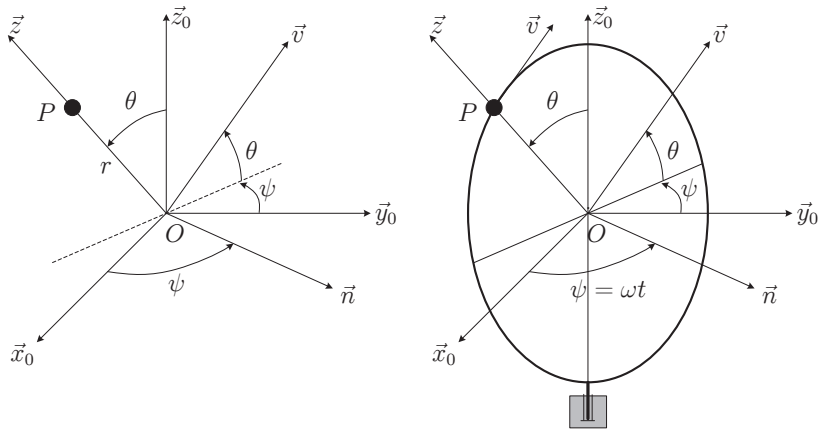


Figure 7.3. *Particle on a hoop rotating about a fixed axis*

- Primitive constraint equations: $\alpha = \omega t$.
- Retained parameters: r, ψ, θ, t. Hence $\overrightarrow{OP} = r\,\vec{z}(\psi, \theta)$. The position of the hoop (C) depends only on t.
- Complementary constraint equation: $r = a$ and $\psi = \alpha = \omega t$.

We have deliberately not taken into account the constraint equations $r = a$ and $\psi = \alpha$, but put them into complementary equations. This is done in order to show how to calculate the compatible VV in the presence of complementary constraint equations and the significance of condition [7.6].

Any VV associated with this parameterization and compatible with the mechanical joint between the particle and the hoop is

$$\vec{V}^*(p) = \frac{\overrightarrow{\partial P}}{\partial r}\dot{r}^* + \frac{\overrightarrow{\partial P}}{\partial \theta}\dot{\theta}^* + \frac{\overrightarrow{\partial P}}{\partial \psi}\dot{\psi}^* = \vec{z}(\psi, \theta)\,\dot{r}^* - r\,\vec{v}(\psi, \theta)\,\dot{\theta}^* + r\sin\theta\,\vec{n}(\psi)\,\dot{\psi}^*$$

with

$$\begin{cases} \dot{r}^* = 0, \quad \dot{\psi}^* = 0 & \text{according to [7.4]} \\ \underline{\text{and } r = a, \quad \psi = \omega t} & \text{according to [7.6]} \end{cases}$$

That is

$$\vec{V}^*(p) = -a\dot{\theta}^*\,\vec{v}(\omega t, \theta) \qquad [7.15]$$

It should be noted that

– the (real) velocity of the particle with respect to R_0 is

$$\vec{V}_{R_0}(p, t) = \frac{\overrightarrow{\partial P}}{\partial r}\dot{r} + \frac{\overrightarrow{\partial P}}{\partial \theta}\dot{\theta} + \frac{\overrightarrow{\partial P}}{\partial \psi}\dot{\psi} + \frac{\overrightarrow{\partial P}}{\partial t} = \dot{r}\,\vec{z} - r\dot{\theta}\,\vec{v} + r\sin\theta\,\dot{\psi}\,\vec{n}$$

– and the (real) velocity *permitted* by the mechanical joint is derived from the previous one by making $r = a$ et $\psi = \omega t$:

$$\vec{V}_{R_0}(p, t) = -a\dot{\theta}\,\vec{v}(\omega t, \theta) + a\omega\sin\theta\,\vec{n}(\omega t)$$

Thus, the (real) velocity *permitted* by the joint and the VV *compatible* with the joint are *different*.

7.2. Invariance of the compatible VVFs with respect to the choice of the primitive parameters

7.2.1. *Context*

Assume that the *a priori* position of the system $\mathcal{S}$ in the reference frame R_0 is defined by N primitive position parameters and, possibly, by time t. Consider two different parameterizations that have the same number $(N+1)$ of primitive parameters: in the first parameterization, we choose $q_1, \ldots, q_N, t$ as primitive parameters, while in the second parameterization, we choose $r_1, \ldots, r_N, t$ as the primitive parameters.

Furthermore, we consider a given mechanical joint in the system, expressed by a certain number of constraint equations. These equations can be written in terms of $(q_1, \ldots, q_N, t)$ or $(r_1, \ldots, r_N, t)$ depending on the chosen parameterization.

The following table describes the two parameterizations studied.

PARAMETERIZATION NO. 1	PARAMETERIZATION NO. 2
Primitive parameters: $q_1, \ldots, q_N$ and t.	Primitive parameters: $r_1, \ldots, r_N$ and t.
Primitive constraint equations: none.	Primitive constraint equations: none.
Retained parameters: the same as the primitive parameters, that is: $q \equiv (q_1, \ldots, q_N), t$. Hence $$P = P(q,t) \quad [7.16]$$	Retained parameters: the same as the primitive parameters, that is: $r \equiv (r_1, \ldots, r_N), t$. Hence $$\widetilde{P} = \widetilde{P}(r,t) \quad [7.17]$$ The function $\widetilde{P}(r,t)$ is not the same as $P(q,t)$ in parameterization no. 1.
Complementary constraint equations: (i) those related to the considered mechanical joint: we decide to write all of them in the differential form and we study the typical equation: $$\sum_{k=1}^{N} \alpha_k(q,t)\dot{q}_k + \beta(q,t) = 0 \quad [7.18]$$ (ii) and those related to other mechanical joints, which do not need to be specified.	Complementary constraint equations: (i) those related to the considered mechanical joint: we decide to write them all in the differential form and the counterpart of the equation of the type [7.18] is written as $$\sum_{h=1}^{N} \widetilde{\alpha}_h(r,t)\dot{r}_h + \widetilde{\beta}(r,t) = 0 \quad [7.19]$$ (ii) and those related to other mechanical joints, which do not need to be specified.

Since the parameters are used to define the position of the current particle of the system, the change in parameters described in the table means that the position of the system is expressed with new parameters. If the parameters q and r are space coordinates (for example, Cartesian or cylindrical coordinates), a change in parameters amounts to changing the coordinates. In the general framework, the parameters q and r may have any nature and are not necessarily space coordinates.

It is assumed that there is a C^1-diffeomorphism (that is, a bijection of class C^1 whose inverse is also of class C^1):

$$\begin{array}{ccc} \text{A subset of } \mathbb{R}^N & \rightarrow & \text{A subset of } \mathbb{R}^N \\ q & \mapsto & r \end{array} \qquad [7.20]$$

In other words, there exists a mapping $q \mapsto r$, which is bijective, of class C^1 and whose Jacobian $\dfrac{Dq}{Dr} \equiv \dfrac{D(q_1, \ldots, q_N)}{D(r_1, \ldots, r_N)}$ is $\neq 0$. Once the parameterizations nos. 1 and 2 are chosen, the bijection $q \mapsto r = r(q)$ or its inverse $r \mapsto q) = q(r)$ are *known*. The bijection [7.20] is schematized in Figure 7.4(a), together with another bijection that we will see later.

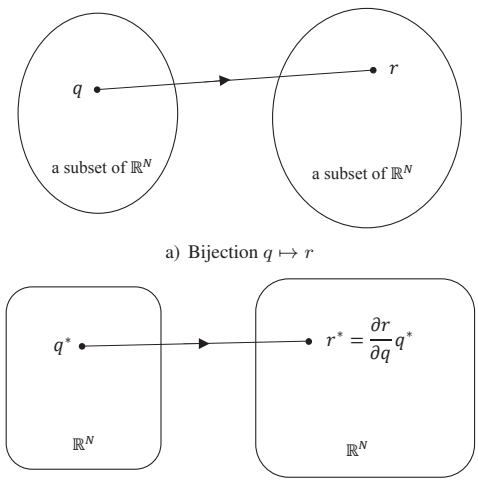


Figure 7.4. *The two bijections* [7.20] *and* [7.36]

There is no use in the previous table showing any possible *resolved* constraint equation related to the considered mechanic joint, whether it is a primitive or a complementary equation. Indeed, if there does exist a *resolved* constraint equation in parameterization no. 1, then its corresponding equation in parameterization no. 2 (obtained by the variable change $q \mapsto r$) is not generally a resolved equation and vice versa. For instance, the resolved constraint equation $r = R$ in cylindrical coordinates becomes $\sqrt{x^2 + y^2} = R$ in Cartesian coordinates and is not resolved. This is why we decided to write all resolved constraint equations in the differential form [7.18] or [7.19] and to treat them as differential constraint equations.

The objective of this section is to show that under certain non-restrictive conditions, the VVFs associated with the previous two parameters and that are compatible with the mechanical joint do *not* depend on the considered parameterizations and are, therefore, *identical*. In order to do this, we will first study the relationships that exist between the real and virtual quantities resulting from the two parameterizations.

7.2.2. *Relationships between the real quantities resulting from the two parameterizations*

The different functions that appear in both parameterizations are different. However, the relationships between them are easy to establish.

Equation [7.17] with parameterization no. 2 is the counterpart of [7.16] from parameterization no. 1. Function $\widetilde{P}(r,t)$, which depends on (r,t), is different from function $P(q,t)$, which depends on (q,t). However, as the two functions give the position of the same particle in R_0, if we replace r in $\widetilde{P}(r,t)$ by $r(q)$, then we must once again arrive at $P(q,t)$:

$$\boxed{P(q,t) = \widetilde{P}(r,t) = \widetilde{P}(r(q),t)} \qquad [7.21]$$

which gives function $P(q,t)$ as a composite function of (q,t). Conversely:

$$\boxed{\widetilde{P}(r,t) = P(q,t) = P(q(r),t)} \qquad [7.22]$$

Relationships $r = r(q)$ or $q = q(r)$ used in [7.21] and [7.22] come from the bijection [7.20], and they are *known* and ready to employ as soon as parameterizations nos. 1 and 2 are chosen.

The complementary constraint equation [7.19] of parameterization no. 2 is the counterpart of [7.18] from parameterization no. 1. Functions $\widetilde{\alpha}_h, \widetilde{\beta}$ do not depend on the same variables as functions α_k, β and they are not identical to functions α_k, β. However, the relationships between these functions are easy to establish: if r in the left-hand side of [7.19] is replaced by $r(q)$, we must arrive at the left-hand side of [7.18]:

$$\boxed{\sum_{k=1}^{N} \alpha_k(q,t)\dot{q}_k + \beta(q,t) = \sum_{h=1}^{N} \widetilde{\alpha}_h(r(q),t)\dot{r}_h + \widetilde{\beta}(r(q),t)} \qquad [7.23]$$

Let us develop the right-hand side of the previous relationship, knowing that $\dot{r}_h = \displaystyle\sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k}\dot{q}_k$:

$$\sum_{h=1}^{N} \widetilde{\alpha}_h(r,t)\dot{r}_h + \widetilde{\beta}(r,t) = \sum_{k=1}^{N} \underbrace{\left(\sum_{h=1}^{N} \widetilde{\alpha}_h(r(q),t)\,\frac{\partial r_h}{\partial q_k}\right)}_{\text{this is } \alpha_k(q,t)} \dot{q}_k + \underbrace{\widetilde{\beta}(r(q),t)}_{\text{this is } \beta(q,t)}$$

Equating this with the left-hand side of [7.18] yields the following relationships between α_k and $\widetilde{\alpha}_k$, on the one hand, and between β and $\widetilde{\beta}$, on the other hand:

$$\boxed{\begin{aligned} \alpha_k(q,t) &= \sum_{h=1}^{N} \widetilde{\alpha}_h(r(q),t)\,\frac{\partial r_h}{\partial q_k}(q) \\ \beta(q,t) &= \widetilde{\beta}(r(q),t) \end{aligned}} \qquad [7.24]$$

A similar reasoning leads to the converse relationships:

$$\boxed{\begin{aligned} \widetilde{\alpha}_h(r,t) &= \sum_{k=1}^{N} \alpha_k(q(r),t)\,\frac{\partial q_k}{\partial r_h}(r) \\ \widetilde{\beta}(r,t) &= \beta(q(r),t) \end{aligned}} \qquad [7.25]$$

7.2.3. *Relationships between the virtual quantities resulting from the two parameterizations*

The VVFs corresponding to the two parameterizations are given by

$$\boxed{\vec{V}^*(p) = \sum_{k=1}^{N} \frac{\overrightarrow{\partial P}}{\partial q_k}(q,t)\,\dot{q}_k^*} \qquad [7.26]$$

$$\boxed{\widetilde{V}^*(p) = \sum_{h=1}^{N} \frac{\widetilde{\partial P}}{\partial r_h}(r,t)\, \dot{r}_h^*} \qquad [7.27]$$

where the N-tuples $(\dot{q}_1^*, \ldots, \dot{q}_N^*)$ and $(\dot{r}_1^*, \ldots, \dot{r}_N^*)$ are *arbitrary and independent*, and where, for easier reading, we have written $\dfrac{\widetilde{\partial P}}{\partial r_h}$ instead of $\dfrac{\overrightarrow{\partial P}}{\partial r_h}$.

The functions $P(q,t)$ and $\widetilde{P}(r,t)$ are connected through [7.21], however, as the tuples $(\dot{q}_1^*, \ldots, \dot{q}_N^*)$ and $(\dot{r}_1^*, \ldots, \dot{r}_N^*)$ are independent of each other; the VVFs $\vec{V}^*(p)$ and $\widetilde{V}^*(p)$ are not, *a priori*, identical. The following theorem gives the relationships between the two VVFs.

Theorem. By taking $r = r(q)$, we have the following relationship between the VVFs resulting from the two parameterization nos. 1 and 2:

$$\boxed{\vec{V}^*(p) = \widetilde{V}^*(p)\big|_{r=r(q)} + \sum_{h=1}^{N} \left(\sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k} \dot{q}_k^* - \dot{r}_h^* \right) \frac{\widetilde{\partial P}}{\partial r_h}(r(q),t)} \qquad [7.28]$$

Conversely, by taking $q = q(r)$:

$$\boxed{\widetilde{V}^*(p) = \vec{V}^*(p)\big|_{q=q(r)} + \sum_{k=1}^{N} \left(\sum_{h=1}^{N} \frac{\partial q_k}{\partial r_h} \dot{r}_h^* - \dot{q}_k^* \right) \frac{\overrightarrow{\partial P}}{\partial q_k}(q(r),t)} \qquad [7.29]$$

PROOF. We only have to prove [7.28]; since the converse relationship [7.29] can then be obtained in the same manner. We have

$$\begin{aligned}
\vec{V}^*(p) &= \sum_{k=1}^{N} \frac{\overrightarrow{\partial P}}{\partial q_k}(q,t)\, \dot{q}_k^* && \text{according to [7.26]} \\
&= \sum_{k=1}^{N} \sum_{h=1}^{N} \frac{\widetilde{\partial P}}{\partial r_h}(r(q),t) \frac{\partial r_h}{\partial q_k} \dot{r}_h^* && \text{because } \frac{\overrightarrow{\partial P}}{\partial q_k} = \sum_{h=1}^{N} \frac{\widetilde{\partial P}}{\partial r_h} \frac{\partial r_h}{\partial q_k} \text{ according to [7.21]} \\
&= \sum_{h=1}^{N} \frac{\widetilde{\partial P}}{\partial r_h}(r(q),t) \left(\sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k} \dot{q}_k^* \right)
\end{aligned}$$

By subtracting the last equation obtained and the equality [7.27], we obtain [7.28]. ■

For later convenience, let us also prove the following relationship:

Theorem. We have

$$\forall (\dot{q}_1^*, \ldots, \dot{q}_N^*), \quad \boxed{\sum_{k=1}^{N} \alpha_k(q,t)\, \dot{q}_k^* = \sum_{h=1}^{N} \widetilde{\alpha}_h(r(q),t) \left(\sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k} \dot{q}_k^* \right)} \qquad [7.30]$$

Conversely,

$$\forall (\dot{r}_1^*, \ldots, \dot{r}_N^*), \quad \boxed{\sum_{h=1}^{N} \widetilde{\alpha}_h(r,t)\, \dot{r}_h^* = \sum_{k=1}^{N} \alpha_k(q(r),t) \left(\sum_{h=1}^{N} \frac{\partial q_k}{\partial r_h} \dot{r}_h^* \right)} \qquad [7.31]$$

PROOF. It is enough to prove [7.30]; the converse relationship [7.31] can then be obtained in the same manner. The summation $\sum_{k=1}^{N}$ on relationship [7.24] weighted by q_k^* gives

$$\sum_{k=1}^{N} \alpha_k \dot{q}_k^* = \sum_{k=1}^{N}\sum_{h=1}^{N} \widetilde{\alpha}_h \frac{\partial r_h}{\partial q_k}\dot{q}_k^* = \sum_{h=1}^{N} \widetilde{\alpha}_h \left(\sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k}\dot{q}_k^*\right) \quad \blacksquare$$

7.2.4. *Identity between the VVFs associated with the two parameterizations and compatible with a mechanical joint*

On applying definition [7.2], we arrive at the VVFs associated with the parameterization nos. 1 and 2 and compatible with the mechanical joint.

COMPATIBLE VVF RESULTING FROM PARAMETERIZATION NO. 1	COMPATIBLE VVF RESULTING FROM PARAMETERIZATION NO. 2
$\vec{V}^*(p) = \sum_{k=1}^{N} \frac{\overrightarrow{\partial P}}{\partial q_k}(q,t)\, \dot{q}_k^*$	$\widetilde{V}^*(p) = \sum_{h=1}^{N} \frac{\widetilde{\partial P}}{\partial r_h}(r,t)\, \dot{r}_h^*$
where $(\dot{q}_1^*, \ldots, \dot{q}_N^*)$ satisfies $$\sum_{k=1}^{N} \alpha_k \dot{q}_k^* = 0 \qquad [7.32]$$	where $(\dot{r}_1^*, \ldots, \dot{r}_N^*)$ satisfies $$\sum_{h=1}^{N} \widetilde{\alpha}_h \dot{r}_h^* = 0 \qquad [7.33]$$
and where q, which appears in the $\frac{\overrightarrow{\partial P}}{\partial q_k}(q,t)$ (expressions obtained *after* differentiation) as well as in the $\alpha_k(q,t)$, satisfies equations [7.18]: $$\sum_{k=1}^{N} \alpha_k(q,t)\dot{q}_k + \beta(q,t) = 0 \qquad [7.34]$$	and where r, which appears in the $\frac{\widetilde{\partial P}}{\partial r_h}(r,t)$ (expressions obtained *after* differentiation) as well as in the $\widetilde{\alpha}_h(r,t)$, satisfies equations [7.19]: $$\sum_{h=1}^{N} \widetilde{\alpha}_h(r,t)\dot{r}_h + \widetilde{\beta}(r,t) = 0 \qquad [7.35]$$

We will construct a new bijection based on bijection [7.20]. For a given q, let us consider the (non-singular) Jacobian matrix $\frac{\partial r}{\partial q}(q)$ of the bijection [7.20], evaluated at q, and define the *linear* mapping

$$\begin{array}{rcl} \mathbb{R}^N & \to & \mathbb{R}^N \\ \dot{q}^* & \mapsto & \dot{r}^* = \frac{\partial r}{\partial q}\dot{q}^*, \end{array} \qquad [7.36]$$

where $\dot{q}^*$ and $\dot{r}^*$ denote the column-vectors with components $(\dot{q}_1^*, \ldots, \dot{q}_N^*)$ and $(\dot{r}_1^*, \ldots, \dot{r}_N^*)$, respectively. Relationship $\dot{r}^* = \frac{\partial r}{\partial q}\dot{q}^*$ signifies that $\dot{r}_h^* = \sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k}\dot{q}_k^*$, $\forall h \in [1, N]$. The mapping [7.36], defined for each fixed q, is linear and, therefore, *bijective*. This is schematized in Figure 7.4(b), with comparison to bijection [7.20].

Theorem. Identity between the compatible VVFs associated with the two parameterizations.

1. The *compatible* VVF resulting from parameterization no. 1 has *the same expression* as the *compatible* VVF resulting from parameterization no. 2, provided that the latter VVF is calculated with $r = r(q)$ and $\dot{r}^* = \dfrac{\partial r}{\partial q}\dot{q}^*$:

$$\boxed{\text{compatible } \vec{V}^*(p) = \text{ compatible } \widetilde{V}^*(p)\Big|_{r\,=\,r(q) \text{ and } \dot{r}^* \,=\, \frac{\partial r}{\partial q}\dot{q}^*}} \qquad [7.37]$$

2. Conversely, the *compatible* VVF resulting from parameterization no. 2 has *the same expression* as the *compatible* VVF resulting from parameterization no. 1, provided that the previous VVF is calculated with $q = q(r)$ and $\dot{q}^* = \dfrac{\partial q}{\partial qr}\dot{r}^*$:

$$\boxed{\text{compatible } \widetilde{V}^*(p) = \text{ compatible } \vec{V}^*(p)\Big|_{q\,=\,q(r) \text{ and } \dot{q}^* \,=\, \frac{\partial q}{\partial r}\dot{r}^*}} \qquad [7.38]$$

3. Consequently, the two *compatible* VVFs, respectively, associated with parameterization nos. 1 and 2, are *identical*, on the condition that they are calculated with $r = r(q)$ and $\dot{r}^* = \dfrac{\partial r}{\partial q}\dot{q}^*$, or, conversely, with $q = q(r)$ and $\dot{q}^* = \dfrac{\partial q}{\partial r}\dot{r}^*$.

   In other words, the compatible VVFs resulting from the two parameterizations whose respective primitive parameters are (q, t) and (r, t) are *identical*, on the condition that certain relationships are imposed between r and q and between $\dot{r}^*$ and $\dot{q}^*$.

   In short, *compatible* VVFs do *not* depend on the choice of the primitive parameters. [7.39]

PROOF. As the two equalities [7.37] and [7.38] are analogous, it is sufficient to prove the first equality. Let $\vec{V}^*(p)$ be a VV resulting from parameterization no. 1 and compatible with the mechanical joint; it is given by [7.26] where $\dot{q}_k^*$ must satisfy [7.32] and where q must satisfy [7.34].

Knowing q and $\dot{q}^*$, let us construct the VV $\widetilde{V}_{R_1}^*(p)$ resulting from parameterization no. 2, expressed by [7.27], such that $r = r(q)$ and $\dot{r}^* = \frac{\partial r}{\partial q}\dot{q}^*$.

– One the one hand, the parameter r satisfies [7.35]. Indeed

$$\sum_{h=1}^{N} \widetilde{\alpha}_h(r(q), t)\, \dot{r}_h + \widetilde{\beta}(r(q), t) \underset{[7.23]}{=} \sum_{k=1}^{N} \alpha_k(q, t)\, \dot{q}_k + \beta(q, t) \underset{[7.34]}{=} 0$$

– On the other hand, the parameter $\dot{r}^*$ satisfies [7.33]. Indeed

$$\sum_{h=1}^{N} \widetilde{\alpha}_h(r(q), t)\, \dot{r}_h^* = \sum_{h=1}^{N} \widetilde{\alpha}_h(r(q), t) \left(\sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k} \dot{q}_k^* \right) \underset{[7.30]}{=} \sum_{k=1}^{N} \alpha_k(q, t)\, \dot{q}_k^* \underset{[7.32]}{=} 0$$

Consequently, the VV $\widetilde{V}_{R_1}^*(p)$ thus constructed is compatible. Moerover, according to [7.28], we have

$$\widetilde{V}^*(p)\Big|_{r=r(q)} = \vec{V}^*(p) \qquad \blacksquare$$

In equality [7.37], the velocity $\vec{V}^*(p)$ is a function of (q,t) and $\dot{q}^*$, while the velocity $\widetilde{V}^*(p)$ is a function of (r,t) and $\dot{r}^*$. In order to express the right-hand side as a function of the same variables (q,t) and $\dot{q}^*$ as the left-hand side:

1. we must replace r by $r = r(q)$. Let us recall that this relationship is known and is imposed as soon as parameterization nos. 1 and 2 have been chosen. This operation is natural, so much so that it may be implicit,

2. we must also replace $\dot{r}^*$ by $\dot{r}^* = \dfrac{\partial r}{\partial q}\dot{q}^*$. Unlike $r = r(q)$, this relationship is not naturally imposed, but is the result of a process of reasoning. It may not be implicit, but must be stated explicitly. A relationship like $\dot{r}^* = \dot{q}^*$ does not allow one to get equality [7.37].

A similar observation can be made regarding the reciprocal equality [7.38].

REMARK. From the proof of the previous theorem, it can be seen that:

– if $\dot{q}^*$ satisfies [7.32], then $\dot{r}^* = \dfrac{\partial r}{\partial q}\dot{q}^*$ satisfies [7.33] evaluated with $r = r(q)$,

– conversely, if $\dot{r}^*$ satisfies [7.33], then $\dot{q}^* = \dfrac{\partial q}{\partial r}\dot{r}^*$ satisfies [7.32] evaluated with $q = q(r)$.

Thus, the mapping $\dot{q}^* \mapsto \dot{r}^* = \dfrac{\partial r}{\partial q}\dot{q}^*$, restricted to the set of $\dot{q}^*$ which satisfies [7.32], is a bijection from this set into the set of $\dot{r}^*$ that satisfies [7.33], (of course, provided that we make $r = r(q)$ in $\widetilde{\alpha}_h(r,t)$ of [7.32], or conversely, $q = q(r)$ in the $\alpha_k(q,t)$ of [7.33]). ■

7.2.5. *Example*

Consider a reference frame R_0 endowed with the orthonormal coordinate system $(O;\vec{x}_0,\vec{y}_0)$ and consider a particle p which is forced to move along a hoop in the plane $O\vec{x}_0\vec{y}_0$. The center of the hoop is O and its radius is R (Figure 7.5). The position of the particle in the reference frame R_0 at a current instant t is denoted by P, with polar coordinates r,θ and Cartesian coordinates x,y.

The mechanical joint imposed on the particle is expressed through the single constraint equation $r = R$, or $\sqrt{x^2+y^2} = R$.

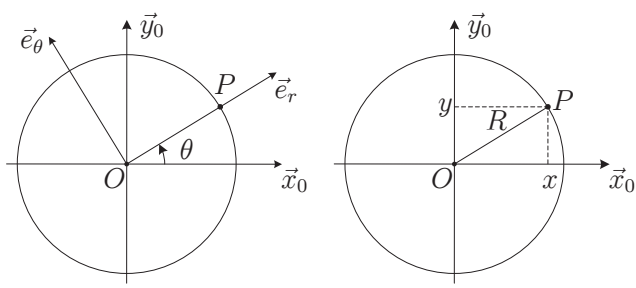


Figure 7.5. *Particule on a hoop*

We will establish the VVs of the particle compatible with the mechanical joint and associated with two distinct parameterizations, one of which has polar coordinates as its primitive parameters, while the other has Cartesian coordinates.

PARAMETERIZATION NO. 1	PARAMETERIZATION NO. 2
Primitive parameters: r, θ.	Primitive parameters: x, y.
Primitive constraint equations: none.	Primitive constraint equations: none.
Retained parameters: the same as the primitive parameters, namely r, θ. Hence $$P(r,\theta) = r\vec{e}_r(\theta)$$	Retained parameters: the same as the primitive parameters, namely x, y. Hence $$\widetilde{P}(x,y) = x\vec{x}_0 + y\vec{y}_0$$
Complementary constraint equation: $r = R$; i.e., in the differential form: $$\dot{r} = 0$$ This equation plays the role of [7.18].	Complementary constraint equation: $\sqrt{x^2+y^2} = R$; i.e., in the differential form: $$x\dot{x} + y\dot{y} = 0$$ This equation plays the role of [7.19].

Using the notations from the previous sections, we have $q = (r, \theta)$ and $r = (x, y)$ (the radius r and the pair $r = (x, y)$ must not be confused!).

A VV associated with parameterization no. 1 and compatible with the mechanical joint between the particle and the hoop is

$$\vec{V}^*(p) = \frac{\overrightarrow{\partial P}}{\partial r}\dot{r}^* + \frac{\overrightarrow{\partial P}}{\partial \theta}\dot{\theta}^* = \dot{r}^*\vec{e}_r(\theta) + r\dot{\theta}^*\vec{e}_\theta(\theta) \qquad \text{where} \begin{cases} \dot{r}^* = 0 \\ \text{and } r \text{ satisfies } \dot{r} = 0 \text{ (or } r = R) \end{cases} \tag{7.40}$$

A VV associated with parameterization no. 2 and compatible with the mechanical joint is

$$\widetilde{V}^*(p) = \frac{\widetilde{\partial P}}{\partial x}\dot{x}^* + \frac{\widetilde{\partial P}}{\partial y}\dot{y}^* = \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0 \quad \text{where} \begin{cases} \dot{x}^*, \dot{y}^* \text{ satisfy } x\dot{x}^* + y\dot{y}^* = 0 \\ \text{and } x, y \text{ satisfy } x\dot{x} + y\dot{y} = 0 \text{ (or } \sqrt{x^2+y^2} = R) \end{cases} \tag{7.41}$$

• Let us start from a VV associated with parameterization no. 1 and compatible with the mechanical joint (see relationship [7.40]). The mapping that represents the change from the polar coordinates to the Cartesian coordinates

$$\begin{array}{ccc} \mathbb{R}_+^* \times]-\pi, \pi[& \to & \mathbb{R}^2 \backslash I \\ (r, \theta) & \mapsto & (x = r\cos\theta, y = r\sin\theta) \end{array} \qquad \text{with } I \equiv \{(x,0) \in \mathbb{R}^2 | x < 0\}$$

is a C^∞-diffeomorphism. Its Jacobian matrix is

$$\frac{\partial r}{\partial q} = \begin{pmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

Consequently, the condition $\dot{r}^* = \dfrac{\partial r}{\partial q}\dot{q}^*$ in [7.37], that is $\forall h \in [1, N]$, $\dot{r}_h^* = \displaystyle\sum_{k=1}^{N} \frac{\partial r_h}{\partial q_k}\dot{q}_k^*$, can be written as

$$\begin{Bmatrix} \dot{x}^* \\ \dot{y}^* \end{Bmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{Bmatrix} \dot{r}^* \\ \dot{\theta}^* \end{Bmatrix} \qquad \text{or} \qquad \left| \begin{array}{l} \dot{x}^* = \cos\theta\,\dot{r}^* - r\sin\theta\,\dot{\theta}^* = -r\sin\theta\,\dot{\theta}^* \\ \dot{y}^* = \sin\theta\,\dot{r}^* + r\cos\theta\,\dot{\theta}^* = \ r\cos\theta\,\dot{\theta}^* \end{array} \right. \tag{7.42}$$

Taking into account the relationships $x = r\cos\theta, y = r\sin\theta$, the pair $(\dot{x}^*, \dot{y}^*)$ given by [7.42] satisfies $x\dot{x}^* + y\dot{y}^* = 0$, whence the VV $\widetilde{V}^*(p) = \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0$ resulting from parameterization no. 2 is compatible. Moreover:

$$\widetilde{V}^*(p)\Big|_{\dot{r}^* = \dfrac{\partial r}{\partial q}\dot{q}^*} = \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0\Big|_{\dot{x}^*, \dot{y}^* \text{ given by [7.42]}} = r\dot{\theta}^*\vec{e}_\theta(\theta) \underset{[7.40]}{=} \vec{V}^*(p)$$

The two compatible VVFs [7.40] and [7.41] are identical, in accordance with [7.37].

• It is also possible to work with the converse relationships. Let us start from a VV associated with parameterization no. 2 and compatible with the mechanical joint (see relationship [7.41]). The inverse Jacobian matrix is

$$\frac{\partial q}{\partial r} = \begin{pmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta/r & \cos\theta/r \end{pmatrix}$$
$$= \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix}$$

Consequently, the condition $\dot{q}^* = \dfrac{\partial q}{\partial r}\dot{r}^*$ in [7.38], that is $\forall k \in [1, N]$, $\dot{q}_k^* = \displaystyle\sum_{h=1}^{N} \frac{\partial q_k}{\partial r_h}\dot{r}_h^*$, can be written as

$$\begin{Bmatrix} \dot{r}^* \\ \dot{\theta}^* \end{Bmatrix} = \begin{pmatrix} \dfrac{x}{\sqrt{x^2+y^2}} & \dfrac{y}{\sqrt{x^2+y^2}} \\ -\dfrac{y}{x^2+y^2} & \dfrac{x}{x^2+y^2} \end{pmatrix} \begin{Bmatrix} \dot{x}^* \\ \dot{y}^* \end{Bmatrix} \quad \text{or} \quad \left| \begin{array}{l} \dot{r}^* = \dfrac{x\dot{x}^* + y\dot{y}^*}{\sqrt{x^2+y^2}} = 0 \\ \dot{\theta}^* = \dfrac{-y\dot{x}^* + x\dot{y}^*}{x^2+y^2} \end{array} \right. \qquad [7.43]$$

Because the $\dot{r}^*$ given by [7.43] satisfies $\dot{r}^* = 0$, the VV $\vec{V}^*(p) = \dot{r}^*\vec{e}_r(\theta) + r\dot{\theta}^*\vec{e}_\theta(\theta)$ resulting from parameterization no. 1 is compatible. Moreover:

$$\vec{V}^*(p)\Big|_{\dot{q}^* = \dfrac{\partial q}{\partial r}\dot{r}^*} = \dot{r}^*\vec{e}_r(\theta) + r\dot{\theta}^*\vec{e}_\theta(\theta)\Big|_{\dot{r}^*, \dot{\theta}^* \text{ given by [7.43]}} = r\frac{-y\dot{x}^* + x\dot{y}^*}{x^2+y^2}\vec{e}_\theta(\theta)$$

Taking into account $r\cos\theta = x, r\sin\theta = y$ and $x\dot{x}^* + y\dot{y}^* = 0$, we arrive at

$$\vec{V}^*(p)\Big|_{\dot{q}^* = \dfrac{\partial q}{\partial r}\dot{r}^*} = \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0 \underset{[7.41]}{=} \widetilde{V}^*(p)$$

The two compatible VVFs [7.41] and [7.40] are identical, in accordance with [7.38].

REMARK. In this section, we considered two parameterizations that have the same number N of primitive position parameters, but do not have the same primitive parameters. While we could also examine the case of two parameterizations that do not have the same number of primitive position parameters, this case offers nothing special, as can be seen through the following simple example.

Let there be a particle moving in the plane $O\vec{x}_0\vec{y}_0$. Two parameterizations can be considered:

– in the first parameterization, we take three Cartesian coordinates x, y, z of the particle as the primitive parameters and we take $z = 0$ as the primitive constraint equation,

– in the second parameterization, we assume $z = 0$ from the start and, consequently, we only take two Cartesian coordinates x, y in the plane as the primitive parameters. With this parameterization, there is no primitive constraint equation.

This situation is summarized in the following table.

PARAMETERIZATION NO. 1	PARAMETERIZATION NO. 2
Primitive parameters: x, y, z.	Primitive parameters: x, y.
Primitive constraint equation: $z = 0$.	Primitive constraint equation: none.
Retained parameters: x, y. Hence $$P(x, y) = x\vec{x}_0 + y\vec{y}_0$$	Retained parameters: x, y. Hence $$P(x, y) = x\vec{x}_0 + y\vec{y}_0$$
Complementary constraint equation: $\cdots$	Complementary constraint equation: $\cdots$

There is no use in specifying the complementary constraint equations. If these do exist, they are *the same* for both parameterizations as they are relationships between the same retained parameters x, y.

Let us now assume that the particle is subjected to a certain mechanical joint in the plane $O\vec{x}_0\vec{y}_0$. The VVF compatible with this joint is *the same* for both parameterizations since it is based on the same *complementary* constraint equations. ■

7.3. Invariance of the compatible VVFs with respect to the choice of the retained parameters

Now that it is known that the compatible VVFs do not change when the primitive parameters are changed, let us fix a set of primitive parameters and study what happens when the retained parameters are changed. We will show that the compatible VVFs also do not change with a change in retained parameters or, in other words, with a change in the classification of constraint equations.

7.3.1. *Context*

Let us assume that the *a priori* position of the system $\mathcal{S}$ in the reference frame R_0 is defined by the primitive parameters $q_1, \ldots, q_N$ and t. As in section 7.1, it is assumed that the constraint equations related to all existing mechanical joints in the system are composed of two types of equations:

1. *resolved* holonomic equations, for instance $q_N = \chi_n(q, t)$, where $q \equiv (q_1, \cdots, q_n)$, $n \leq N$,

2. and non-holonomic constraint equations in differential form $\displaystyle\sum_{i=1}^{n} \alpha_i(q, t)\, \dot{q}_i + \beta(q, t) = 0$.

   The unresolved holonomic constraint equations of the type $f(q, t) = 0$, if they exist, are systematically recast in the differential form.

A resolved holonomic constraint equation may be classified as a primitive or complementary equation depending on the user's choice. If it is written as a primitive equation, this signifies that it is used to eliminate the resolved parameter. On the contrary, an unresolved, holonomic or non-holonomic constraint equation must necessarily be written as a complementary equation, since it cannot be used to eliminate a position parameter.

Consider a given mechanical joint in the system, expressed by a certain number of constraint equations. On the other hand, we consider two parameterizations that differ in the placement of a single constraint equation *taken from among those related to the mechanical joint under consideration*:

– in the first parameterization, the constraint equation in question is classified as primitive (it is, therefore, necessarily a *resolved holonomic* equation),

– while the second parameterization is classified as complementary.

The following table describes both of these parameterizations; the constraint equation [7.44], which changes its place from one parameterization to the other is written in a box in order to highlight it.

PARAMETERIZATION NO. 1 (this is the usual parameterization considered in [2.19])	**PARAMETERIZATION NO. 2**
Primitive parameters: $\underbrace{q_1, \ldots, q_n}_{\equiv q}, q_{n+1}, \ldots, q_N$ and t.	Primitive parameters: the same as in parameterization no. 1.
Primitive constraint equations: it is assumed that there exist the following primitive constraint equations: $$\boxed{q_{n+1} = \chi_{n+1}(q,t)} \quad [7.44]$$ $$q_{n+2} = \chi_{n+2}(q,t)$$ $$\vdots$$ $$q_N = \chi_N(q,t)$$ where Equation [7.44] is specific to the considered mechanical joint, the other equations come from all existing mechanical joints in the system.	Primitive constraint equations: the same as in parameterization no. 1, except for Equation [7.44] which is moved to complementary equations: $$q_{n+2} = \chi_{n+2}(q,t)$$ $$\vdots$$ $$q_N = \chi_N(q,t)$$
Retained parameters: $q \equiv (q_1, \ldots, q_n), t$. Hence $$P = P(q,t) \quad [7.45]$$	Retained parameters: $\hat{q} \equiv (q_1, \ldots, q_n, q_{n+1}), t$. Compared to parameterization no. 1, there is the additional parameter q_{n+1}. Hence $$P = \hat{P}(\hat{q},t) \quad [7.46]$$ The function $\hat{P}(\hat{q},t)$ is not the same as $P(q,t)$ from parameterization no. 1.
Complementary constraint equations: (i) those related to the considered mechanical joint: these may or may not be resolved holonomic equations. We will study two typical equations: $$q_n = \chi_n(q_1, \ldots, q_{n-1}, t) \quad [7.47]$$ $$\sum_{i=1}^{n} \alpha_i(q,t)\, \dot{q}_i + \beta(q,t) = 0 \quad [7.48]$$ (ii) and those related to other mechanical joints, which do not need to be specified.	Complementary constraint equations: (i) those related to the mechanical joint under consideration: $$\boxed{q_{n+1} = \chi_{n+1}(q,t)}$$ $$q_n = \chi_n(q_1, \ldots, q_{n-1}, t)$$ $$\sum_{i=1}^{n+1} \hat{\alpha}_i(\hat{q},t)\, \dot{q}_i + \hat{\beta}(\hat{q},t) = 0 \quad [7.49]$$ (ii) and those related to other mechanical joints, which do not need to be specified.

The objective of this section is to show that the VVFs associated with the two parameterizations given above and which are compatible with the mechanical joint do *not*, in fact, depend on the parameterization being considered and are, therefore, *identical*. In order to show this, we will first study the relationships that exist between the real and virtual quantities resulting from both these parameterizations, respectively.

REMARK. One has only to study the case described in the above table, where we move a single equation from parameterization no. 1 to parameterization no. 2. The general case, where several equations are moved, can be treated by repeating the above case several times in succession, moving one equation at a time. ■

7.3.2. Relationships between the real quantities resulting from the two parameterizations

Equation [7.46] in parameterization no. 2 is the counterpart of [7.45] from parameterization no. 1, by adopting the point of view of the user who chooses parameterization no. 2. The function $\hat{P}(\hat{q},t)$, which depends on $(\hat{q},t)$ is different from the function $P(q,t)$, which depends on (q,t). However, the relationship between the two functions is quite simple to establish, knowing that they give the position of the same particle in R_0: if we replace q_{n+1} in $\hat{P}(\hat{q},t)$ by $q_{n+1} = \chi_{n+1}(q,t)$, we must once again arrive at $P(q,t)$:

$$\boxed{P(q,t) = \hat{P}(\hat{q},t) = \hat{P}(q,\chi_{n+1}(q,t),t)}, \qquad [7.50]$$

which gives function $P(q,t)$ as a composite function of (q,t).

Equation [7.47] from parameterization no. 1 typically designates a resolved holonomic constraint equation that may be classified as a primitive equation (to remove q_n from the retained parameters) but which we decide to classify as a complementary equation. This equation is written as is in parameterization no. 2. Here, unlike what we did for [7.46], there is no need to take into account $q_{n+1} = \chi_{n+1}(q,t)$ simply because q_{n+1} does not appear in [7.47].

The non-holonomic complementary constraint equation [7.49] from parameterization no. 2 is the counterpart of [7.48] from parameterization no. 1, by adopting the point of view of the user who chooses parameterization no. 2. The functions $\hat{\alpha}_i, \hat{\beta}$ are not identical to the functions α_i, β as they are written in terms of the retained parameters, which are not the same in parameterization no. 1. However, the relationships between these functions can be easily established: if we replace q_{n+1} in the left-hand side of [7.49] with $q_{n+1} = \chi_{n+1}(q,t)$, then we must once again arrive at the left-hand side of [7.48]:

$$\boxed{\sum_{i=1}^{n} \alpha_i(q,t)\,\dot{q}_i + \beta(q,t) = \sum_{i=1}^{n+1} \hat{\alpha}_i(q,\chi_{n+1}(q,t),t)\,\dot{q}_i + \hat{\beta}(q,\chi_{n+1}(q,t),t)} \qquad [7.51]$$

Let us develop the right-hand side of the previous relationship:

$$\sum_{i=1}^{n+1} \hat{\alpha}_i(\hat{q},t)\,\dot{q}_i + \hat{\beta}(\hat{q},t) = \sum_{i=1}^{n} \hat{\alpha}_i(\hat{q},t)\,\dot{q}_i + \hat{\alpha}_{n+1}(\hat{q},t)\,\dot{q}_{n+1} + \hat{\beta}(\hat{q},t)$$

where $\dot{q}_{n+1} = \sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i}\, \dot{q}_i + \frac{\partial \chi_{n+1}}{\partial t}$. Hence

$$\sum_{i=1}^{n+1} \hat{\alpha}_i(\hat{q},t)\, \dot{q}_i + \hat{\beta}(\hat{q},t) = \sum_{i=1}^{n} \Big(\hat{\alpha}_i(\hat{q},t) + \hat{\alpha}_{n+1}(\hat{q},t) \underbrace{\frac{\partial \chi_{n+1}}{\partial q_i}}_{\text{function of } (q,t)} \Big)\, \dot{q}_i + \hat{\beta}(\hat{q},t) + \hat{\alpha}_{n+1}(\hat{q},t) \underbrace{\frac{\partial \chi_{n+1}}{\partial t}}_{\text{function of}(q,t)}$$

By equating this with the right-hand side of [7.51], we arrive at the following relationships between α_i and $\hat{\alpha}_i$, on the one hand, and between β and $\hat{\beta}$, on the other hand:

$$\boxed{\begin{aligned} \alpha_i(q,t) &= \hat{\alpha}_i(q, \chi_{n+1}(q,t), t) + \hat{\alpha}_{n+1}(q, \chi_{n+1}(q,t), t) \frac{\partial \chi_{n+1}}{\partial q_i}(q,t) \\ \beta(q,t) &= \hat{\beta}(q, \chi_{n+1}(q,t), t) + \hat{\alpha}_{n+1}(q, \chi_{n+1}(q,t), t) \frac{\partial \chi_{n+1}}{\partial t}(q,t) \end{aligned}} \qquad [7.52]$$

which gives the functions α_i, β as composite functions of (q,t).

7.3.3. *Relationships between the virtual quantities resulting from the two parameterizations*

The VVFs corresponding to the two parameterizations, respectively, are given by

$$\boxed{\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i}(q,t)\, \dot{q}_i^*} \qquad [7.53]$$

$$\boxed{\hat{V}^\sharp(p) = \sum_{i=1}^{n+1} \frac{\widehat{\partial P}}{\partial q_i}(\hat{q},t)\, \dot{q}_i^\sharp} \qquad [7.54]$$

where the n-tuple $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ and the $(n+1)$-tuple $(\dot{q}_1^\sharp, \ldots, \dot{q}_{n+1}^\sharp)$ (these two tuples do not have the same number of elements) are *arbitrary and independent*.

The functions $P(q,t)$ and $\hat{P}(\hat{q},t)$ are related through [7.50]. However, as the two tuples $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ and $(\dot{q}_1^\sharp, \ldots, \dot{q}_{n+1}^\sharp)$ are independent of each other, the VVFs $\vec{V}^*(p)$ and $\hat{V}^\sharp(p)$ are not, *a priori*, identical.

We will establish the relationship between the two VVFs using the relationships between the $\dot{q}_i^*$ and the $\dot{q}_i^\sharp$.

Definition. The VVF associated with parameterization no. 2 and *twinned with parameterization no. 1*, is, by definition, the VVF [7.54] where $(\dot{q}_1^\sharp, \ldots, \dot{q}_n^\sharp)$ *is taken to be equal to* $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ *of the VVF* [7.53]:

$$\boxed{\hat{V}^\sharp(p) = \sum_{i=1}^{n} \frac{\widehat{\partial P}}{\partial q_i}(\hat{q},t)\, \dot{q}_i^* + \frac{\widehat{\partial P}}{\partial q_{n+1}}(\hat{q},t)\, \dot{q}_{n+1}^\sharp} \qquad [7.55]$$

(there is no condition on $\dot{q}_{n+1}^\sharp$, which is an arbitrary scalar specific to parameterization no. 2). In other words, the two VVFs are twinned when they have *the same virtual* $\dot{q}_j^*$ for all j indices *common* to the two parameterizations.

We also say that the VVFs $\vec{V}^*(p)$ and $\vec{V}^\sharp(p)$ are *twinned*.

Do not say that the virtual parameters $\dot{q}_i^*$ and $\dot{q}_i^\sharp$ are equal as only some of them are equal.

Theorem. Relationship between twinned VVFs.

(i) The *twinned* VVs are related through

$$\boxed{\vec{V}^*(p) = \hat{V}^\sharp(p)\big|_{q_{n+1}=\chi_{n+1}(q,t)} + \left(\sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i}\dot{q}_i^* - \dot{q}_{n+1}^\sharp\right) \frac{\widehat{\partial P}}{\partial q_{n+1}}(q, \chi_{n+1}(q,t), t)} \quad [7.56]$$

(ii) In a rigid body S, the *twinned* virtual angular velocities are related through

$$\boxed{\vec{\Omega}^*_{R_1S}(p) = \hat{\Omega}^\sharp_{R_1S}(p)\big|_{q_{n+1}=\chi_{n+1}(q,t)} + \left(\sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i}\dot{q}_i^* - \dot{q}_{n+1}^\sharp\right) \hat{\omega}^{n+1}_{R_1S}\big|_{q_{n+1}=\chi_{n+1}(q,t)}} \quad [7.57]$$

where $\hat{\omega}^{n+1}_{R_1S}$ is the partial angular velocity of S with respect to R_1 defined by [2.36] (with $n+1$ parameters, instead of n).

PROOF.

(i) We have

$$\begin{aligned}
\vec{V}^*(p) &= \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i}(q,t)\,\dot{q}_i^* \qquad \text{according to [7.53]} \\
&= \sum_{i=1}^{n} \left(\frac{\widehat{\partial P}}{\partial q_i}(q, \chi_{n+1}(q,t), t) + \frac{\widehat{\partial P}}{\partial q_{n+1}}(q, \chi_{n+1}(q,t), t)\frac{\partial \chi_{n+1}}{\partial q_i}(q,t)\right)\dot{q}_i^* \\
&\quad \text{on applying [7.50]} \\
&= \sum_{i=1}^{n} \frac{\widehat{\partial P}}{\partial q_i}(q, \chi_{n+1}(q,t), t)\,\dot{q}_i^* + \frac{\widehat{\partial P}}{\partial q_{n+1}}(q, \chi_{n+1}(q,t), t)\sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i}(q,t)\,\dot{q}_i^*
\end{aligned}$$

Hence relationship [7.56], noting that according to [7.55] the first term in the last right-hand side of the equation is written as

$$\sum_{i=1}^{n} \frac{\widehat{\partial P}}{\partial q_i}(q, \chi_{n+1}(q,t), t)\,\dot{q}_i^* = \hat{V}^\sharp(p)\big|_{q_{n+1}=\chi_{n+1}(q,t)} - \frac{\widehat{\partial P}}{\partial q_{n+1}}(q, \chi_{n+1}(q,t), t)\,\dot{q}_{n+1}^\sharp$$

(ii) To prove [7.57], let us apply [7.56] to two different particles p, p' of the rigid body S, whose respective positions in R_0 are P and P', and then subtract the obtained relationships:

$$\underbrace{\vec{V}^*(p') - \vec{V}^*(p)}_{\underset{[4.32]}{=} \vec{\Omega}^*_{R_1S}\times\overrightarrow{PP'}} = \underbrace{\hat{V}^\sharp(p') - \hat{V}^\sharp(p)}_{\underset{[4.32]}{=} \hat{\Omega}^\sharp_{R_1S}\times\overrightarrow{PP'}} + \left(\sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i}\dot{q}_i^* - \dot{q}_{N+1}^\sharp\right) \underbrace{\left(\frac{\widehat{\partial P'}}{\partial q_{n+1}} - \frac{\widehat{\partial P}}{\partial q_{n+1}}\right)}_{\underset{[2.42]}{=} \hat{\omega}^{n+1}_{R_1S}\times\overrightarrow{PP'}}$$

where $q_{n+1} = \chi_{n+1}(q,t)$ is implied in the right-hand side of the equation, for brevity. Note that [2.42] is applicable because hypothesis [2.33], assumed to be verified in this chapter, encompasses hypothesis [2.26]. We arrive at relationship [7.57] by taking into account the fact that $\overrightarrow{PP'}$ is arbitrary. ■

For later convenience, let us also prove the following relationship:

Theorem. $\forall(\dot{q}_1^*, \ldots, \dot{q}_n^*)$, $\forall \dot{q}_{n+1}^\sharp$ (here, we do *not* require twinned VVFs),

$$\boxed{\sum_{i=1}^{n} \alpha_i \, \dot{q}_i^* = \sum_{i=1}^{n} \hat{\alpha}_i \, \dot{q}_i^* + \hat{\alpha}_{n+1} \, \dot{q}_{n+1}^\sharp + \hat{\alpha}_{n+1} \left(\sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i} \dot{q}_i^* - \dot{q}_{n+1}^\sharp \right)} \qquad [7.58]$$

where the functions $\hat{\alpha}_i, i \in [1, n+1]$ on the right-hand side of the equation are evaluated at $(q, \chi_{n+1}(q,t), t)$.

PROOF. The proof is simple: the summation $\sum_{i=1}^{n}$ over relationship [7.52], weighted by q_i^*, gives

$$\sum_{i=1}^{n} \alpha_i q_i^* = \sum_{i=1}^{n} \hat{\alpha}_i q_i^* + \hat{\alpha}_{n+1} \sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i} q_i^*$$

By adding $\hat{\alpha}_{n+1} \dot{q}_{n+1}^\sharp$ to the last term and by subtracting the same term, we arrive at the desired result. Q.E.D. ■

7.3.4. *Identity between the VVFs associated with the two paramaterizations and compatible with a mechanical joint*

On applying definition [7.2], we arrive at the VVFs associated with parameterization nos. 1 and 2, and compatible with the mechanical joint:

COMPATIBLE VVF RESULTING FROM PARAMETERIZATION NO. 1	COMPATIBLE VVF RESULTING FROM PARAMETERIZATION NO. 2
$\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i}(q,t) \, \dot{q}_i^*$	$\hat{V}^\sharp(p) = \sum_{i=1}^{n+1} \frac{\widehat{\partial P}}{\partial q_i}(\hat{q},t) \, \dot{q}_i^\sharp$
where $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ satisfies	where $(\dot{q}_1^\sharp, \ldots, \dot{q}_{n+1}^\sharp)$ satisfies $\dot{q}_{n+1}^\sharp = \sum_{i=1}^{n} \frac{\partial \chi_{n+1}}{\partial q_i} \dot{q}_i^\sharp$ [7.59]
$\dot{q}_n^* = \sum_{i=1}^{n-1} \frac{\partial \chi_n}{\partial q_i} \dot{q}_i^*$ [7.60]	$\dot{q}_n^\sharp = \sum_{i=1}^{n-1} \frac{\partial \chi_n}{\partial q_i} \dot{q}_i^\sharp$ [7.61]
$\sum_{i=1}^{n} \alpha_i(q,t) \dot{q}_i^* = 0$ [7.62]	$\sum_{i=1}^{n+1} \hat{\alpha}_i(\hat{q},t) \dot{q}_i^\sharp = 0$ [7.63]

and where q, involved in the $\frac{\partial \overrightarrow{P}}{\partial q_i}(q,t)$, the $\frac{\partial \chi_n}{\partial q_i}$ (expressions obtained *after* differentiation) as well as in the $\alpha_i(q,t)$, satisfies $$q_n = \chi_n(q_1,\ldots,q_{n-1},t)$$ $$\sum_{i=1}^{n} \alpha_i(q,t)\,\dot{q}_i + \beta(q,t) = 0 \qquad [7.64]$$	and where $\hat{q}$, involved in the $\frac{\partial \overrightarrow{P}}{\partial q_i}(\hat{q},t)$, the $\frac{\partial \chi_n}{\partial q_i}$ (expressions obtained *after* differentiation) as well as in the $\hat{\alpha}_i(\hat{q},t)$, satisfies $$q_n = \chi_n(q_1,\ldots,q_{n-1},t)$$ $$q_{n+1} = \chi_{n+1}(q,t)$$ $$\sum_{i=1}^{n+1} \hat{\alpha}_i(\hat{q},t)\,\dot{q}_i + \hat{\beta}(\hat{q},t) = 0 \qquad [7.65]$$

Qualitatively speaking, parameterization no. 2 has more retained parameters and the associated VVF is, therefore, richer: it depends on $(\dot{q}_1^\sharp,\ldots,\dot{q}_n^\sharp)$ *and* $\dot{q}_{n+1}^\sharp$. However, as it is subjected to more restrictions, we may expect to obtain the same VVF as with parameterization no. 1. This is what will now be proved.

Theorem. Identity between the compatible VVFs associated with two parameterizations. The two *compatibles* VVFs, resulting from parameterization nos. 1 and 2 are *twinned*. Stronger still – they are *identical*:

$$\boxed{\text{compatible } \vec{V}^*(p) = \text{ compatible } \hat{V}^\sharp(p)} \qquad [7.66]$$

Recall that the *compatible* VVFs do not change when the classification of constraint equation is changed. In other words, the *compatible* VVFs do *not* depend on the choice of retained parameters.

PROOF.

(i) Let us first show that the solutions of [7.64] are the same as those of [7.65]. More specifically, let us show that

$$\begin{cases} \text{– if } q \text{ satisfies [7.64], then } q \text{ and } q_{n+1} = \chi_{n+1}(q,t) \text{ satisfy [7.65],} \\ \text{– conversely, if } q \text{ and } q_{n+1} \text{ satisfy [7.65], then } q \text{ satisfies [7.64].} \end{cases} \qquad [7.67]$$

As the above statement is evident for relationships $[7.64]_1$ and $[7.65]_1$, which are identical, let us show it for solutions of $[7.64]_2$ and $[7.65]_{2-3}$. In order to do this, one only has to recall relationship [7.51]:

$$\sum_{i=1}^{n} \alpha_i(q,t)\,\dot{q}_i + \beta(q,t) = \sum_{i=1}^{n+1} \hat{\alpha}_i(q,\chi_{n+1}(q,t),t)\,\dot{q}_i + \hat{\beta}(q,\chi_{n+1}(q,t),t)$$

Hence, the statement [7.67].

(ii) Let us now show that the solutions of [7.60] and [7.62] are the same as those of [7.59], [7.61] and [7.63]. More specifically, let us show that

$$\begin{cases} \text{– if } (\dot{q}_1^*,\ldots,\dot{q}_n^*) \text{ satisfies [7.60] and [7.62], then } (\dot{q}_1^\sharp,\ldots,\dot{q}_n^\sharp) = (\dot{q}_1^*,\ldots,\dot{q}_n^*) \text{ and an} \\ \text{arbitrary } \dot{q}_{n+1}^\sharp \text{ satisfies [7.59], [7.61] and [7.63],} \\ \text{– conversely, if } (\dot{q}_1^\sharp,\ldots,\dot{q}_n^\sharp,\dot{q}_{n+1}^\sharp) \text{ satisfies [7.59], [7.61] and [7.63], then} \\ (\dot{q}_1^*,\ldots,\dot{q}_n^*) = (\dot{q}_1^\sharp,\ldots,\dot{q}_n^\sharp) \text{ satisfies [7.60] and [7.62].} \end{cases} \qquad [7.68]$$

Indeed:

- as equation [7.59] exists only in parameterization no. 2 (it has no equivalent in parameterization no. 1), there is nothing to be proven for this equation,
- as equations [7.60] and [7.61] are identical when $*$ and $\sharp$ are interchanged, any $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ that satisfies [7.60] gives $(\dot{q}_1^\sharp, \ldots, \dot{q}_n^\sharp) = (\dot{q}_1^*, \ldots, \dot{q}_n^*)$ satisfying [7.61] and vice versa,
- what remains now is to consider the solutions of [7.62] and [7.63]. Taking into account [7.59] and [7.65], relationship [7.58] becomes: $\forall(\dot{q}_1^*, \ldots, \dot{q}_n^*)$, $\forall \dot{q}_{n+1}^\sharp$,

$$\sum_{i=1}^{n} \alpha_i(q,t)\ \dot{q}_i^* = \sum_{i=1}^{n} \hat{\alpha}_i(q, \chi_{n+1}(q,t), t)\ \dot{q}_i^* + \hat{\alpha}_{n+1}(q, \chi_{n+1}(q,t), t)\ \dot{q}_{n+1}^\sharp$$

  Consequently:

  - $*$ if $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ satisfies [7.62], then $(\dot{q}_1^\sharp, \ldots, \dot{q}_n^\sharp) = (\dot{q}_1^*, \ldots, \dot{q}_n^*)$ and an arbitrary $\dot{q}_{n+1}^\sharp$ satisfy relationship [7.63] calculated with [7.65],
  - $*$ conversely, if $(\dot{q}_1^\sharp, \ldots, \dot{q}_n^\sharp, \dot{q}_{n+1}^\sharp)$ satisfies [7.63] calculated with [7.65], then $(\dot{q}_1^*, \ldots, \dot{q}_n^*) = (\dot{q}_1^\sharp, \ldots, \dot{q}_n^\sharp)$ satisfies [7.62].

We have thus proven statement [7.68]. Consequently, the two compatible VVFs resulting from the two parameterizations, respectively, are *twinned*.

(iii) As the two VVFs are twinned, it is possible to apply [7.56]. Taking into account [7.59], relationship [7.56] becomes [7.66]. ■

7.3.5. *Example 1*

We will verify theorem [7.66] through a few simple examples. Let us return to the example in section 7.1.3 for a particle p moving along a curve in the plane $O\vec{x}_0\vec{y}_0$, with equation $y = \chi(x)$ (Figure 7.1). This time, let us calculate the VVs of the particle that are compatible with the mechanical joint and associated with the following parameterization, where the constraint equation $y = \chi(x)$ is classified as primitive:

REDUCED PARAMETERIZATION. [7.69]

- Primitive parameters: the same as in the total parameterization: x, y.
- Primitive constraint equation: $y = \chi(x)$.
- Retained parameter: x. Hence $P = P(x) = O + x\vec{x}_0 + \chi(x)\vec{y}_0$.
- Complementary constraint equation: none.

According to definition [7.2], a VV associated with this parameterization and compatible with the mechanized joint is

$$\vec{V}^*(p) = \frac{\overrightarrow{\partial P}}{\partial x}\dot{x}^* = \left[\vec{x}_0 + \frac{d\chi}{dx}(x)\vec{y}_0\right]\dot{x}^* \qquad [7.70]$$

As predicted by [7.66], this VV is identical to the VV [7.11] found in section 7.1.3 and resulting from the total parameterization.

7.3.6. *Example 2*

Resume the example in section 7.1.4 of a particle p moving along a curve in the plane $O\vec{x}_0\vec{y}_0$, with the polar equation $r = \chi(\theta)$ (Figure 7.2). This time, let us calculate the VVs of the particle that are compatible with the mechanical joint and associated with the following parameterization, where the constraint equation $r = \chi(\theta)$ is classified as primitive:

REDUCED PARAMETERIZATION. [7.71]

- Primitive parameters: the same as in the total parameterization: r, θ.
- Primitive constraint equation: $r = \chi(\theta)$.
- Retained parameter: θ . Hence $P = O + r\vec{e}_r(\theta) = O + \chi(\theta)\vec{e}_r(\theta) = P(\theta)$.
- Complementary constraint equation: none.

According to definition [7.2], a VV associated with this parameterization and compatible with the mechanical joint is

$$\vec{V}^*(p) = \frac{\overrightarrow{\partial P}}{\partial \theta}\dot{\theta}^* = \left[\frac{d\chi}{d\theta}(\theta)\ \vec{e}_r(\theta) + \chi(\theta)\ \vec{e}_\theta(\theta)\right]\dot{\theta}^* \qquad [7.72]$$

As predicted by [7.66], this VV is identical to the VV [7.13] found in section 7.1.4 and resulting from the total parameterization.

REMARK. This is where we understand the significance of condition [7.6] in the definition of a VVF compatible with a joint. With the total parameterization considered in section 7.1.4, condition [7.6] forces r to satisfy the complementary constraint equation $r = \chi(\theta)$ and it leads to expression [7.13] for the compatible VV, identical to [7.72].

If we had not made $r = \chi(\theta)$, we would have obtained the following expression for the VV compatible with the particle, instead of [7.13]:

$$\vec{V}^*(p) = \left[\frac{d\chi}{d\theta}(\theta)\ \vec{e}_r(\theta) + r\ \vec{e}_\theta(\theta)\right]\dot{\theta}^*, \qquad [7.73]$$

which is a different expression from [7.72]. ■

7.3.7. *Example 3: particle moving along a hoop rotating around a fixed axis*

Let us now return to the example of the particle p moving along a hoop (C) rotating at a constant rate about a fixed axis, considered in section 7.1.5. Recall that the primitive parameter of (C) is α, the primitive parameters of p are r, ψ, θ and that the mechanical joint between the particle and the hoop is expressed by the *resolved* constraint equations $r = a$ and $\psi = \alpha$ (i.e. $\vec{n} = \vec{c}$). Let us calculate the VVs of the particle compatible with the mechanical joint and associated with two parameterizations that are different from the parameterization considered in section 7.1.5.

• The first parameterization is as follows:

REDUCED PARAMETERIZATION. [7.74]

- Primitive parameters: r, ψ, θ and α.
- Primitive constraint equations: $\alpha = \omega t$ and $r = a$.

- Retained parameters: ψ, θ, t. Hence, $\overrightarrow{OP} = a\ \vec{z}(\psi, \theta)$. The position of the hoop (C) depends only on t.
- Complementary constraint equation: $\psi = \alpha = \omega t$.

A VV associated with this reduced parameterization and compatible with the mechanical joint is

$$\vec{V}^*(p) = \frac{\partial \overrightarrow{P}}{\partial \theta}\dot{\theta}^* + \frac{\partial \overrightarrow{P}}{\partial \psi}\dot{\psi}^* = -a\dot{\theta}^*\ \vec{v}(\psi, \theta) + a \sin\theta\ \dot{\psi}^*\ \vec{n}(\psi) \text{ with } \begin{cases} \dot{\psi}^* = 0 & \text{according to [7.4]} \\ \underline{\text{and } \psi = \omega t} & \text{according to [7.6]} \end{cases}$$

That is

$$\vec{V}^*(p) = -a\dot{\theta}^*\ \vec{v}(\omega t, \theta) \qquad [7.75]$$

• The second parameterization is as follows:

INDEPENDENT PARAMETERIZATION. [7.76]

- Primitive parameters: r, ψ, θ and α.
- Primitive constraint equations: $\alpha = \omega t$, $r = a$ and $\psi = \alpha = \omega t$.
- Retained parameters: θ, t. Hence $\overrightarrow{OP} = a\ \vec{z}(\omega t, \theta)$. The position of the hoop (C) depends only on t.
- Complementary constraint equation: none.

A VV associated with this independent parameterization and compatible with the mechanical joint is

$$\vec{V}^*(p) = \frac{\partial \overrightarrow{P}}{\partial \theta}\dot{\theta}^* = -a\dot{\theta}^*\ \vec{v}(\omega t, \theta) \qquad [7.77]$$

As predicted by [7.66], the VVs [7.75] and [7.77] are identical to the VV [7.15] found in section 7.1.5 and resulting from the total parameterization.

REMARK. It is seen that, at a given instant t, one can derive the VV of the particle p compatible with the joint, from the real velocity as follows:

– to imagine that *the hoop is fixed in the position in R_0, that it occupies at the considered instant t*,

– to calculate the velocity that particle p would have at this position: $\vec{V}(p, t) = -a\dot{\theta}\ \vec{v}(\omega t, \theta)$,

– to then derive the VV $\vec{V}^*(p)$ by formally replacing $\dot{\theta}$ with $\dot{\theta}^*$. ■

7.4. Invariance of the compatible VVFs with respect to the choice of the parameterization

Let us summarize the principal facts established since the beginning of this chapter. Consider a system $\mathcal{S}$ made up of several rigid bodies and subjected to a certain number of mechanical joints. The *a priori* position of system $\mathcal{S}$ in the reference frame R_0 is defined by a certain number of primitive position parameters chosen beforehand, and possibly the time t. The existing mechanical joints are expressed by a certain number of constraint equations. Choosing a

parameterization consists of classifying these equations in a certain manner – some as primitive and others as complementary, which amounts to choosing the retained parameters.

We are concerned with a given mechanical joint in this system, and in [7.2] we define the concept of the VVF associated with a chosen parameterization and *compatible* with the mechanical joint.

In the previous sections, we proved two interesting results:

1. The first of these is theoreom [7.39], according to which the *compatible* VVFs do *not* depend on the choice of the primitive parameters. Owing to this result, we can fix a set of primitive parameters and then study what happens when we change the retained parameters.

2. The second is theorem [7.66], according to which the *compatible* VVFs do *not* depend on the choice of the retained parameters.

Now one only has to combine these two results to arrive at the following conclusion:

Theorem. [7.78]

The VVF *compatible* with a given mechanical joint is *independent* of the choice of the primitive parameters and the retained parameters. In other words, it is independent of the choice of the parameterization. The VVF *compatible* with the considered joint *has the same expression*, whatever parameterization we choose. *It depends only on the considered mechanical joint*.

In order to express these properties, we say that the concept of the compatible VVF defined by [7.2] is an *intrinsic* concept. We also say that the compatible VVF is *invariant* with respect to the choice of the parameterization.

Thus, from now on we will simply speak of a *VVF compatible with a mechanical joint*, without needing to specify the parameterization that we are working with.

7.5. Perfect joints

7.5.1. *Definition of a perfect joint*

A mechanical joint in the system $\mathcal{S}$ may be of one of the following types:

1. either an *internal* joint; i.e. between two rigid bodies S_1 and S_2 of the system $\mathcal{S}$ (Figure 7.6(a)),

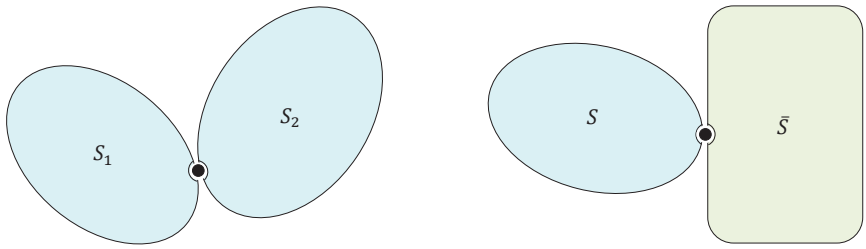
2. or a joint *with the exterior*; i.e. between a rigid body S of the system and a rigid body $\bar{S}$, which does not belong to the system ($\bar{S}$ may be fixed or moving in R_1) (Figure 7.6(b)).

Let us examine the VP done by the constraint efforts at this mechanical joint, using the summary given in section 5.6:

1. In the case of an internal mechanical joint between two rigid bodies S_1 and S_2 in the system $\mathcal{S}$, the efforts exerted by the joint on the system $\mathcal{S}$ are the inter-efforts $\mathcal{F}_{S_1 \leftrightarrow S_2}$ between these rigid bodies. The VP of the constraint efforts is that of the inter-efforts and according to theorem [5.14], it is *independent of the reference frame* with respect to which it is calculated. It is denoted $\mathscr{P}^*(\mathcal{F}_{S_1 \leftrightarrow S_2}, t)$ *without the reference frame index*. Let us recall its expression:

$$\mathscr{P}^*(\mathcal{F}_{S_1 \leftrightarrow S_2}, t) = \mathcal{M}_{S_2 \to S_1}(t) \circ V^*_{S_2 S_1}$$

The VP of the inter-efforts only depends on the moment field of the efforts $\mathcal{M}_{S_2 \to S_1}(t)$ and the *relative* virtual velocities $\vec{V}^*_{S_1 S_2}$ between two rigid bodies.



a) Internal joint between two rigid bodies S_1 and S_2 of the system

b) Joint between a rigid body S of the system and a rigid body $\bar{S}$ outside the system

Figure 7.6. *Mechanical joints in a system*

2. In the case of a mechanical joint between a rigid body S of the system and a rigid body $\bar{S}$ outside the system, we require hypothesis [2.33] adopted at the beginning of this chapter:

   HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{01}$ of R_1 with respect to R_0 and the point O_1 fixed in R_1 do *not* depend on q.

   The VP of the constraint efforts is that of the efforts exerted by rigid body $\bar{S}$ on rigid body S. Using the previous hypothesis, we showed in [5.8] and [5.11] that this VP is *independent of the reference frame* R_1 with respect to which it is calculated. This enables us to write it *without the reference frame index*: $\mathscr{P}^*(\mathcal{F}_{\bar{S}\to S}, t)$. Its expression follows from [5.5]:

$$\mathscr{P}^*(\mathcal{F}_{\bar{S}\to S}, t) = \mathcal{M}_{\bar{S}\to S}(t) \circ V^*_{R_1 S}$$

   being aware that here the VVF $V^*_{R_1 S}$ is, in fact, independent of R_1.

In a generic manner, the VP of the constraint efforts exerted by the joint in question on system $\mathcal{S}$ is denoted by $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}, t)$. Depending on whether this involves an internal joint or a joint with the exterior, this notation denotes either the VP of inter-efforts $\mathscr{P}^*(\mathcal{F}_{S_1\leftrightarrow S_2}, t)$, or the VP $\mathscr{P}^*(\mathcal{F}_{\bar{S}\to S}, t)$ of an external rigid body on a rigid body in $\mathcal{S}$.

Because of the notation $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}, t)$, we can introduce the following definition:

Definition. A mechanical joint in the system $\mathcal{S}$ is said to be *perfect* (or *ideal*) if, at any instant t, the VP of the constraint efforts exerted by the joint on $\mathcal{S}$ is *zero* in *any VVF* V^* *compatible* with this joint:

$$\forall t, \forall \text{ VVF } V^* \textit{ compatible} \text{ with this joint}, \boxed{\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}, t) = 0} \qquad [7.79]$$

In other words, the mechanical joint is *perfect* if, at any instant, any VVF *compatible* with the joint is *orthogonal* (in the sense of the product $\circ$ defined in [5.5]) to the moment field of constraint efforts exerted by the joint in question.

As just seen above, the notion of a perfect joint is independent of the reference with respect to which the VP is calculated. Let us now examine the independence with respect to the chosen parameterization. According to theorem [7.78], the VVFs *compatible* with the considered mechanical joint are *independent* of the choice of parameterization. As a result, the VP of the constraint efforts in a VVF *compatible* with the considered joint is also *independent* of the chosen parameterization (let us recall that the VP is defined as a sum of the products of the efforts and the VVF). In particular, if the VP of the constraint efforts is zero with one parameterization, then it is also zero with another parameterization.

In analytical mechanics, the concept of a perfect joint is, therefore, *intrinsic* in the sense that it does not depend on the chosen parameterization. In Newtonian mechanics, a perfect joint is defined without any mention of the parameterization and, therefore, the question of independence with respect to the parameterization does not arise at all.

In practice, in order to carry out the calculations, we have yet to choose a certain parameterization, calculate the VVFs compatible with this parameterization and, finally, calculate the VP of constraint efforts in these compatible VVFs. When working with this parameterization, it should be remembered that

– the constraint equations that come into play via the compatible VVFs are only *complementary* equations,

– and that the VP is the VP of *all* constraint efforts ensuring the mechanical joint in question (these efforts correspond to the primitive constraint equations *as well as* to the complementary constraint equations).

• Let us assume that the mechanical joint being studied is an internal joint. In the calculation of the VP $\mathscr{P}^*(\mathcal{F}_{S_1 \leftrightarrow S_2}, t)$ of the inter-efforts, we must take *the most general* relative VVF $\vec{V}^*_{S_1 S_2}$, compatible with the mechanical joint in question, and only this joint.

The relative VVF $\vec{V}^*_{S_1 S_2}$ is automatically the most general if the two rigid bodies S_1 and S_2 are subjected to a single mechanical joint between them. On the contrary, if the two rigid bodies are subjected to several joints between them or with other rigid bodies around them, then we must consider releasing the joints other than the joint in question (i.e. considering the two rigid bodies as being solely connected by the joint in question) in order to have the most general possible relative VVF $\vec{V}^*_{S_1 S_2}$.

For instance, assume that the rigid bodies S_1, S_2 are connected by two mechanical joints called A and B and that we wish to study the perfection of joint A. We must then *release joint B and write the compatible VVF corresponding to the system subjected only to joint A*. Releasing a joint means removing all the existing constraint equations that express this joint and, consequently, possibly increasing the number of retained parameters, or even creating new primitive parameters (thus, increasing the number of initial degrees of freedom of the system). The compatible VVF resulting from this operation is not the same as that resulting from the initial parameterization and constitutes *the most general possible* VVF compatible with the joint A.

• The same operation must be carried out if the mechanical joint studied is a joint with the exterior. When calculating the VP $\mathscr{P}^*(\mathcal{F}_{\bar{S} \leftrightarrow S}, t)$ of the constraint efforts, we must take the *most general* VVF $\vec{V}^*_{R_1 S}$ compatible with the mechanical joint in question, and only this.

This condition is automatically satisfied if the rigid body S is subjected to a single mechanical joint with $\bar{S}$. On the other hand, if S is subjected to several joints with other rigid bodies around it, we must consider freeing joints other than the joint in question to satisfy this condition. The VVF $\vec{V}^*_{R_1 S}$ (here, independent of R_1) does indeed depend on the mechanical joints taken into account. This remark will be illustrated in section 7.6 where we deal with a combined joint.

• In the sequel, we will work on some examples of perfect joints. The reader may also consult Appendix 2, which is dedicated to the study of the elementary perfect joints (spherical joint, cylindrical joint, etc.) that are frequently encountered in mechanics.

7.5.2. Example 1

We return to the example of the particle that moves along a planar curve with the equation $y = \chi(x)$ in Cartesian coordinates, discussed in section 7.1.3.

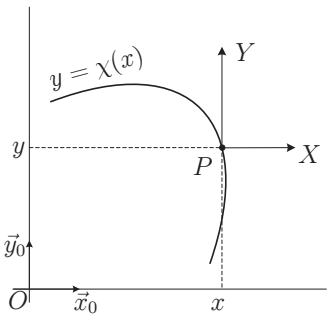


Figure 7.7. *Perfect joint between a particle and a planar curve – parameterization in Cartesian coordinates*

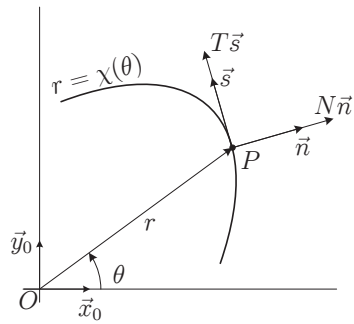


Figure 7.8. *Perfect joint between a particle and a planar curve – parameterization in polar coordinates*

The mechanical joint studied here is the (external) joint imposed by the materialized curve on the particle. The VV compatible with the joint is given by [7.11] or [7.70]: $\vec{V}^*(p) = \left[\vec{x}_0 + \frac{d\chi}{dx}(x)\vec{y}_0\right]\dot{x}^*$.

Let us denote the constraint force exerted by the curve on the particle p by: $\vec{F} = X\vec{x}_0 + Y\vec{y}_0$ (Figure 7.7). The VP of the constraint forces in a VV compatible with the joint is

$$\mathscr{P}^* = \vec{F}.\vec{V}^*(p) = \left[X + Y\frac{d\chi}{dx}(x)\right]\dot{x}^*$$

Definition [7.79] gives:

$$\begin{aligned}
\text{The joint is perfect} \quad &\Leftrightarrow \mathscr{P}^* = 0 \text{ for any compatible VV, i.e. for any } \dot{x}^* \\
&\Leftrightarrow X + Y\frac{d\chi}{dx}(x) = 0 \\
&\Leftrightarrow \text{the constraint force } \vec{F} \text{ is orthogonal to the curve } y = \chi(x) \\
&\Leftrightarrow \text{there is no friction along the curve}
\end{aligned}$$

Through this example, it can be seen that the definition of a perfect joint may be used in two ways:

– One may assume that the constraint efforts satisfy certain properties and then prove that the mechanical joint is perfect.

– Conversely, one may assume, *a priori*, that the joint is perfect and that from this we derive the properties that these constraint efforts must satisfy.

  In a general problem, the hypothesis of a perfect joint provides some *a priori* information about the constraint efforts (this is the same in Newtonian mechanics).

7.5.3. *Example 2*

We return to the example of the particle moving along a planar curve $r = \chi(\theta)$ in polar coordinates, discussed in section 7.1.4.

The VV compatible with the joint is given by [7.13] or [7.72]:

$$\vec{V}^*(p) = \left[\frac{d\chi}{d\theta}(\theta)\,\vec{e}_r(\theta) + \chi(\theta)\,\vec{e}_\theta(\theta)\right]\dot{\theta}^*$$

Let us denote the constraint force exerted on the particle p by $\vec{F} = N\vec{n} + T\vec{s}$, Figure 7.8, where N is the normal force, T is the tangential force, $\vec{s}$ the unit vector tangent to the curve:

$$\vec{s} = \frac{\overrightarrow{dP}/d\theta}{\|\overrightarrow{dP}/d\theta\|} = \frac{1}{\sqrt{\left(\frac{d\chi}{d\theta}\right)^2 + \chi^2(\theta)}}\left[\frac{d\chi}{d\theta}\vec{e}_r + \chi(\theta)\vec{e}_\theta\right]$$

and $\vec{n}$ is the unit vector normal to the curve, defined as

$$\vec{n} = \vec{s} \times \vec{z}_0 = \frac{1}{\sqrt{\left(\frac{d\chi}{d\theta}\right)^2 + \chi^2(\theta)}}\left[-\chi(\theta)\vec{e}_r + \frac{d\chi}{d\theta}\vec{e}_\theta\right]$$

The VP of the constraint forces in a VV compatible with the joint is

$$\mathscr{P}^* = \frac{1}{\sqrt{\left(\frac{d\chi}{d\theta}\right)^2 + \chi^2(\theta)}} T\left[\chi^2(\theta) + \left(\frac{d\chi}{d\theta}\right)^2\right]\dot{\theta}^* \qquad [7.80]$$

Definition [7.79] gives:

$$\begin{aligned}\text{The joint is perfect} \;&\Leftrightarrow\; \mathscr{P}^* = 0 \text{ for any compatible VV i.e. for any } \dot{\theta}^* \\ &\Leftrightarrow\; T = 0 \\ &\Leftrightarrow\; \text{the constraint force } \vec{F} \text{ is orthogonal to the curve } r = \chi(\theta) \\ &\Leftrightarrow\; \text{there is no friction along the curve}\end{aligned}$$

As expected, we arrive at the same final result as in the previous example.

REMARK. Let us continue with the remark made after [7.72] on the significance of condition [7.6] in the definition of a VVF compatible with a constraint. When working with the total parameterization in section 7.1.4, if we had not made $r = \chi(\theta)$, we would have obtained expression [7.73] for the compatible VV and the VP of the constraint force would be

$$\mathscr{P}^* = \vec{F}.\vec{V}^*(p) = \frac{1}{\sqrt{\left(\frac{d\chi}{d\theta}\right)^2 + \chi^2(\theta)}}\left\{N\frac{d\chi}{d\theta}\left[r - \chi(\theta)\right] + T\left[r\chi(\theta) + \left(\frac{d\chi}{d\theta}\right)^2\right]\right\}\dot{\theta}^*$$

In the case of a perfect joint, we *cannot* derive from the above expression that $T = 0$, which is not consistent with what may be expected in physics. Condition [7.6] is, therefore, an important condition. ■

7.5.4. *Example 3*

We return to the example of the particle moving along a hoop rotating about a fixed axis, discussed in section 7.1.5, except that, for the sake of generality, the angular velocity of the hoop is no longer prescribed.

Recall that the studied system is composed of the particle p and hoop (C). The mechanical joint studied here is, therefore, an internal joint. Let $\vec{F}_{C\to p} = N\vec{n} + T\vec{v} + Z\vec{z}$ denote the constraint force exerted by the hoop (C) on particle p, where T is the tangential force, and N and Z are the normal forces (Figure 7.9). Furthermore, when R_C denotes the reference frame defined by (C), the VP of the constraint inter-forces in a VV $\vec{V}^*_{R_C}(p)$ compatible with the joint is calculated using the general formula [5.14]:

$$\mathscr{P}^*(\mathcal{F}_{C\leftrightarrow p}, t) = \vec{F}_{C\to p}.\vec{V}^*_{R_C}(p) \qquad [7.81]$$

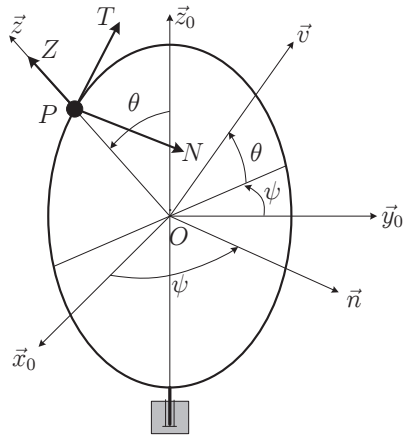


Figure 7.9. *Perfect joint between a particle and a hoop*

The relative VV $\vec{V}^*_{R_C}(p)$ of p with respect to (C) is calculated by the composition formula for velocities [4.43]:

$$\vec{V}^*_{R_C}(p) = \vec{V}^*_{R_0}(p) - \vec{V}^*_{R_0 R_C}(P) \qquad [7.82]$$

where $\vec{V}^*_{R_0 R_C}(P)$ is the VV, with respect to R_0, of the particle of the hoop C, which at instant t coincides with the position $P(t)$ of p (this is the so-called *background virtual velocity* defined in [4.49]). Note, in passing, the relevance or irrelevance of the reference frame indices:

– Using the below parameterization, the reference frame R_C does not satisfy hypothesis [2.33] adopted at the beginning of this chapter. Consequently, the VV $\vec{V}^*_{R_C}(p)$ depends on R_C and the reference frame index R_C cannot be removed.

– On the contrary, as the reference frame R_0 automatically satisfies hypothesis [2.33], the VVs $\vec{V}^*_{R_0}(p)$ and $\vec{V}^*_{R_0 C}(P)$, which appear in the right-hand side of [7.82] do not, in fact, depend on R_0. The reference frame R_0 may be replaced by any other reference frame R_1 (satisfying hypothesis [2.33]) without changing the final result.

To calculate the VVs $\vec{V}^*_{R_0}(p)$ and $\vec{V}^*_{R_0 R_C}(P)$, we choose the following parameterization for the hoop and particle system:

REDUCED PARAMETERIZATION.

- Primitive parameters: r, ψ, θ and α.
- Primitive constraint equation: $\alpha = \psi$.
- Retained parameters: r, ψ, θ. Hence $\overrightarrow{OP} = r\ \vec{z}(\psi, \theta)$. The position of the hoop (C) is defined by ψ.
- Complementary constraint equation: $r = a$.

A VV $\vec{V}^*_{R_0}(p)$ associated with this parameterization and compatible with the mechanical joint between the particle and the hoop is given as

$$\vec{V}^*_{R_0}(p) = \frac{\overrightarrow{\partial P}}{\partial r}\dot{r}^* + \frac{\overrightarrow{\partial P}}{\partial \theta}\dot{\theta}^* + \frac{\overrightarrow{\partial P}}{\partial \psi}\dot{\psi}^* = \vec{z}(\psi, \theta)\ \dot{r}^* - r\ \vec{v}(\psi, \theta)\ \dot{\theta}^* + r \sin\theta\ \vec{n}(\psi)\ \dot{\psi}^*$$

with

$$\begin{cases} \dot{r}^* = 0 & \text{according to [7.4]} \\ \underline{\text{and } r = a} & \text{according to [7.6]} \end{cases}$$

Or

$$\vec{V}^*_{R_0}(p) = -a\dot{\theta}^* \, \vec{v} + a \sin\theta \, \dot{\psi}^* \, \vec{n} \qquad [7.83]$$

The background virtual velocity $\vec{V}^*_{R_0C}(P)$ is calculated using relationship [4.35]:

$$\begin{aligned} \vec{V}^*_{R_0R_C}(P) &= \vec{V}^*_{R_0R_C}(O) + \vec{\Omega}^*_{R_0R_C} \times \overrightarrow{OP} \quad \text{where } \vec{V}^*_{R_0R_C}(O) = \vec{0} \text{ and } \vec{\Omega}^*_{R_0R_C} = \dot{\psi}^* \vec{z}_0 \\ &= a \sin\theta \, \dot{\psi}^* \, \vec{n} \end{aligned} \qquad [7.84]$$

This VV is compatible with the mechanical joint. Inserting relationships [7.83] and [7.84] in [7.82] yields the compatible VV:

$$\vec{V}^*_{R_C}(p) = -a\dot{\theta}^* \, \vec{v} \qquad [7.85]$$

Hence, by [7.81]:

$$\mathscr{P}^*(\mathcal{F}_{C\leftrightarrow p}, t) = \vec{F}_{C\to p}.\vec{V}^*_{R_C}(p) = -aT\dot{\theta}^*$$

Finally, definition [7.79] gives:

$$\begin{aligned} \text{The joint is perfect} \;&\Leftrightarrow\; \mathscr{P}^* = 0 \text{ for any compatible VV i.e. for any } \dot{\theta}^* \\ &\Leftrightarrow\; T = 0 \\ &\Leftrightarrow\; \text{there is no friction between the particle and the hoop} \end{aligned}$$

REMARKS.

1. Another way of obtaining [7.85] is to calculate the VV $\vec{V}^*_{R_C}(p)$ using the general definition [7.9]:

$$\vec{V}^*_{R_C}(p) = \bar{\bar{Q}}_{0C}.\sum_{i=1}^{n} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{C0}.\overrightarrow{OP}\right)\dot{q}_i^* = \bar{\bar{Q}}_{0C}.\left(\frac{\partial}{\partial r}\left(\bar{\bar{Q}}_{C0}.\overrightarrow{OP}\right)\dot{r}^* + \frac{\partial}{\partial \theta}\left(\bar{\bar{Q}}_{C0}.\overrightarrow{OP}\right)\dot{\theta}^*\right) \qquad [7.86]$$

   where the rotation represented by $\bar{\bar{Q}}_{0C}$ is taken to be equal to the rotation by angle ψ about $\vec{z}_0$, which brings $\vec{x}_0$ to $\vec{n}$. The rotation tensor $\bar{\bar{Q}}_{0C}$ thus depends on the parameter ψ and does not satisfy hypothesis [2.33].

   In Figure 7.10, we have depicted the canonical basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ of $\mathbb{R}^3$, as well as the vectors denoted by $\vec{v}^{(C)} \equiv \bar{\bar{Q}}_{C0}.\vec{v}$, $\vec{z}^{(C)} \equiv \bar{\bar{Q}}_{C0}.\vec{z}$, images of $\vec{v}$, $\vec{z}$ in the reference frame R_C. The vector $\bar{\bar{Q}}_{C0}.\overrightarrow{OP}$ is the image of $\overrightarrow{OP}$ in the reference frame R_C: $\bar{\bar{Q}}_{C0}.\overrightarrow{OP} = r\vec{z}^{(C)}$.

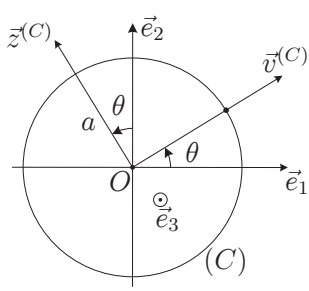


Figure 7.10. *The relative virtual velocity of p with respect to* (C)

Hence

$$\frac{\partial}{\partial r}\left(\bar{\bar{Q}}_{C0}.\overrightarrow{OP}\right)\dot{r}^* + \frac{\partial}{\partial \theta}\left(\bar{\bar{Q}}_{C0}.\overrightarrow{OP}\right)\dot{\theta}^* = \dot{r}^*\vec{z}^{(C)} - r\dot{\theta}^*\vec{v}^{(C)}$$

which, when inserted in [7.86], gives

$$\vec{V}^*_{R_C}(p) = \dot{r}^*\vec{z} - r\dot{\theta}^*\vec{v}$$

The VV compatible with the joint can be derived from this by making $\dot{r}^* = 0$ and $r = a$. We then arrive again at [7.85].

2. If we consider that the system is only made up of the particle p without the hoop, then the joint between the particle and the hoop becomes external to the system and it is *not* perfect. Indeed, the VP of the constraint forces exerted by the hoop on the particle writes

$$\mathscr{P}^*(\mathcal{F}_{C\to p}, t) = \vec{F}_{C\to p}.V^*_{R_1}(p)$$

knowing that the VV $V^*_{R_1}(p)$ does, in fact, have the same expression for any R_1 satisfying hypothesis [2.33]. The VV compatible with the joint may be obtained by reusing [7.83]:

$$\vec{V}^*_{R_1}(p) = \vec{V}^*_{R_0}(p) = -a\dot{\theta}^*\ \vec{v} + a\sin\theta\ \dot{\psi}^*\ \vec{n}$$

Hence

$$\mathscr{P}^*(\mathcal{F}_{C\to p}, t) = (N\vec{n} + T\vec{v} + Z\vec{z}).\vec{V}^*_{R_1}(p) = Na\sin\theta\ \dot{\psi}^* - aT\dot{\theta}^*$$

Definition [7.79] gives:

$$\begin{aligned}\text{The joint is perfect} \ &\Leftrightarrow\ \mathscr{P}^* = 0 \text{ for any compatible VV,}\\ &\qquad\quad \text{i.e. for any } \dot{\psi}^*, \dot{\theta}^*\\ &\Leftrightarrow\ N = T = 0\end{aligned}$$

Contrary to the condition $T = 0$, the condition $N = 0$ is not, in general, feasible and consequently the considered joint is not perfect. Thus, the same mechanical joint may or may not be perfect depending on whether it is an internal joint or an external joint. ■

7.5.5. *Example 4*

The previous examples dealing with the joint between a particle and another rigid body showed that the joint is perfect if and only if the contact is frictionless. The example we now study deals with the joint between two solid rigid bodies and exhibits a new condition: the joint is perfect if the contact between the rigid bodies takes place without friction *or* slipping.

The system studied is a disc S with center C and radius R, moving in the reference frame R_0 endowed with the coordinate system $(O; \vec{x}_0, \vec{y}_0)$, and remaining in contact at point I with the materialized axis $O\vec{x}_0$ (Figure 7.11). The *a priori* position of the disc in R_0 is defined by the coordinates (x, y) of the center C and the angle of rotation φ of the disc, defined as being the angle between $\vec{x}_0$ and a radius $\overrightarrow{CA}$ attached to the disc.

The mechanical joint studied here is the contact at I imposed by the axis $O\vec{x}_0$ on the disc S (it is an external joint). The contact force exerted by the axis on the disc is written, in the general case, as $\vec{F}_{axis\to S} = T\vec{x}_0 + N\vec{y}_0$, where T, N denote, respectively, the tangential and the normal contact force. We will study the perfect character of the joint by distinguishing between two cases: simple contact and contact without slipping.

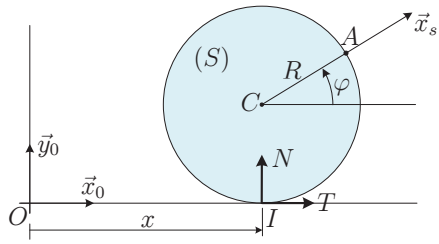


Figure 7.11. *Perfect joint between a disc and an axis*

7.5.5.1. *Case of simple contact at I*

Let us first assume that the contact between the disc and the axis $O\vec{x}_0$ is simple, in the sense that it is expressed only by the geometric relationship $y = R$ and no other additional relationship such as contact without slipping. We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: x, y, φ.
- Primitive constraint equation: $y = R$.
- Retained parameters: x, φ.
- No complementary constraint equation.

The VV at the point C is

$$\begin{aligned}\vec{V}^*_{0S}(C) &= \frac{\partial \overrightarrow{OC}}{\partial x}\dot{x}^* + \frac{\partial \overrightarrow{OC}}{\partial \varphi}\dot{\varphi}^* \quad \text{with } \overrightarrow{OC} = x\vec{x}_0 + R\vec{y}_0 \\ &= \dot{x}^*\vec{x}_0\end{aligned}$$

Hence, the VV at the point I, on applying [4.35], is given as:

$$\vec{V}^*_{0S}(I) = \vec{V}^*_{0S}(C) + \vec{\Omega}^*_{0S} \times \overrightarrow{CI} = (\dot{x}^* + R\dot{\varphi}^*)\vec{x}_0$$

This VV is compatible with the mechanical joint. Relationships [5.5] and [5.8] give the VP of the constraint forces exerted by the axis $O\vec{x}_0$ on the disc S in a VV compatible with the joint:

$$\mathscr{P}^*(\vec{F}_{axis\ \to S}, t) = \vec{F}_{axis\ \to S}.\vec{V}^*_{0S}(I) = T(\dot{x}^* + R\dot{\varphi}^*)$$

(to apply [5.8], we note that the reference frame R_0 automatically satisfies hypothesis [2.33]).

Definition [7.79] gives:

$$\begin{aligned}\text{The joint is perfect} \quad &\Leftrightarrow \mathscr{P}^* = 0 \text{ for any compatible VV, i.e. for any } \dot{x}^*, \dot{\varphi}^* \\ &\Leftrightarrow T = 0 \\ &\Leftrightarrow \text{ there is no friction between the axis } O\vec{x}_0 \text{ and the disc}\end{aligned}$$

7.5.5.2. *Case of no-slip contact at I*

Let us now assume that the contact between the disc and the axis $O\vec{x}_0$ takes place without slipping. The joint at I is therefore expressed through two equations: the geometric relationship $y = R$ and $\vec{V}_{0S}(I) = \vec{0}$, i.e. the semi-holonomic condition $\dot{x} + R\dot{\varphi} = 0$. We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: x, y, φ.
- Primitive constraint equation: $y = R$.
- Retained parameters: x, φ.
- Complementary constraint equation: $\dot{x} + R\dot{\varphi} = 0$.

This time, the VV at the point I, compatible with the mechanical joint is

$$\begin{aligned}\vec{V}^*_{0S}(I) &= \vec{V}^*_{0S}(C) + \vec{\Omega}^*_{0S} \times \overrightarrow{CI} = (\dot{x}^* + R\dot{\varphi}^*)\vec{x}_0 \qquad \text{with } \dot{x}^* + R\dot{\varphi}^* = 0 \text{ according to [7.5]} \\ &= \vec{0}\end{aligned}$$

Hence, the VP of the constraint forces exerted by the axis $O\vec{x}_0$ on the disc S, in a VV compatible with the joint:

$$\mathscr{P}^*(\vec{F}_{axis \to S}, t) = \vec{F}_{axis \to S}.\vec{V}^*_{0S}(I) = 0$$

which means that, according to definition [7.79], the joint at I is perfect.

7.5.5.3. *To summarize*

The results given earlier show that the point contact between the disc and the axis $O\vec{x}_0$ is a perfect joint if the contact takes place *without friction or slipping*. This observation is the same in Newtonian mechanics, where the concept of a perfect joint is defined in a slightly different manner. It remains valid in the following cases (by neglecting the resistance to rolling and pivoting):

- (external) point contact between a rigid body and another rigid body that is at rest in a reference frame R_1 that satisfies hypothesis [2.33],
- (internal) point contact between two rigid bodies in the same system.

7.6. Example: a perfect compound joint

We work in the reference frame R_0 endowed with the coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and we consider a rigid body S made up (i) of a rod AC of length e, parallel to $\vec{z}_S$, and (ii) of a disc of radius a, center C, axis $A\vec{z}_S$ and (iii) of a part of any form, represented schematically in Figure 7.12 by a cylinder.

We define the three Euler angles ψ, θ, φ by taking

- the vector $\vec{n}$ equal to one of the vectors orienting the line of intersection between the plane of the disc and the plane $O\vec{x}_0\vec{y}_0$,
- the precession angle ψ equal to the angle between $\vec{x}_0$ and $\vec{n}$, measured around $\vec{z}_0$,
- the nutation angle θ equal to the angle between $\vec{z}_0$ and $\vec{z}_S$, measured around $\vec{n}$,
- the spin angle φ, which gives the rotation of S about the axis $A\vec{z}_S$.

The intermediate bases $(\vec{n}, \vec{u}, \vec{z}_0)$ and $(\vec{n}, \vec{v}, \vec{z}_S)$ are represented in Figure 7.12.

The *a priori* position of the rigid body S in R_0 is defined by three Cartesian coordinates x, y, z of the point A and three Euler angles ψ, θ, φ.

We study here *the compound joint* imposed by the fixed support on the rigid body S, defined by the combination of two elementary joints:

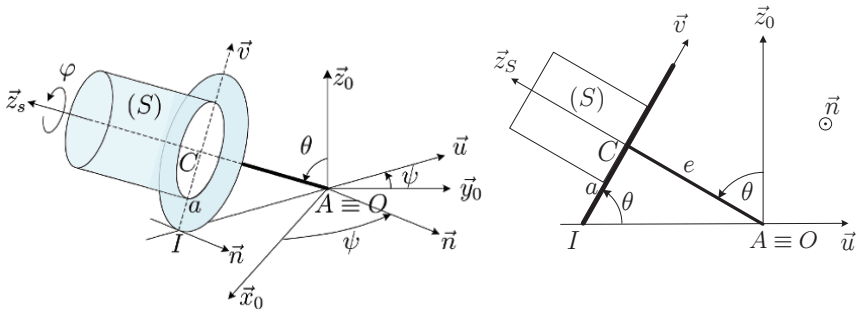


Figure 7.12. *Combination of a ball-and-socket joint and a point contact*

– a ball-and-socket joint that maintains the end A of the shaft AC at the origin O at any instant,

– and a point contact of the disc with the plane $O\vec{x}_0\vec{y}_0$ at the point I.

The constraint efforts exerted by the support S_0 (defining the reference frame R_0) on S are those given as follows:

– at the ball-and-socket joint at A: a force denoted by $\vec{F}_A$ and a torque denoted by $\vec{C}_A$,

– at the contact point I: a contact force $\vec{F}_I$.

We will study the perfect character of the compound joint in two different ways: (i) by considering it as a single global joint and (ii) by considering it as the superimposition of two elementary joints.

7.6.1. *Perfect combined joint*

We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: $x, y, z, \psi, \theta, \varphi$.
- Primitive constraint equations: these are those expressing the spherical joint, namely: $x = y = z = 0$.
- Retained parameters : ψ, θ, φ.
- Complementary constraint equation: the equation expressing the contact at I, namely $z_I = 0$. To express this equation in terms of the retained parameters, let us write:

$$\overrightarrow{AI} = \overrightarrow{AC} + \overrightarrow{CI} = e\vec{z}_S - a\vec{v} = (e\cos\theta - a\sin\theta)\vec{z}_0 - (e\sin\theta + a\cos\theta)\vec{u} \qquad [7.87]$$

Hence, through projection onto $\vec{z}_0$, the new expression for $z_I = 0$:

$$e\cos\theta - a\sin\theta = 0 \qquad \Leftrightarrow \qquad \theta = \theta_0 \equiv \arctan\frac{e}{a}\ (const) \qquad [7.88]$$

The VP of the constraint efforts exerted by the support on the rigid body S is calculated using [5.5] and [5.8]:

$$\mathscr{P}^*(\mathcal{F}_{0\to S},t) = \underbrace{\vec{F}_A.\vec{V}^*_{0S}(A) + \vec{C}_A.\vec{\Omega}^*_{0S}}_{\text{due to the spherical joint}} + \underbrace{\vec{F}_I.\vec{V}^*_{0S}(I)}_{\text{due to the contact at } I} \qquad [7.89]$$

Let us calculate the VVs appearing in the above expression, compatible with the joint:

– $\vec{V}^*_{0S}(A) = \dfrac{\partial\overrightarrow{OA}}{\partial\psi}\dot{\psi}^* + \dfrac{\partial\overrightarrow{OA}}{\partial\varphi}\dot{\varphi}^* = \vec{0}$ because $\overrightarrow{OA} = \vec{0}$.

– On the other hand:

$$\begin{aligned}\vec{\Omega}^*_{0S} &= \dot{\psi}^*\vec{z}_0 + \dot{\theta}^*\vec{n} + \dot{\varphi}^*\vec{z}_S \text{ where } \dot{\theta}^* = 0 \text{ according to [7.88]}\\ &= \dot{\psi}^*\vec{z}_0 + \dot{\varphi}^*\vec{z}_S\end{aligned}$$

– We calculate $\vec{V}^*_{0S}(I)$ by $\vec{V}^*_{0S}(I) = \vec{V}^*_{0S}(A) + \vec{\Omega}^*_{0S} \times \overrightarrow{AI}$, where $\overrightarrow{AI} = -(e\sin\theta + a\cos\theta)\vec{u} = -\dfrac{a}{\cos\theta_0}\vec{u}$, knowing that, according to [7.88], $e\sin\theta + a\cos\theta = \dfrac{a}{\cos\theta_0}$. It is because of clause [7.6] that we were able to use the complementary constraint equation [7.88] to construct the compatible VV. Hence

$$\vec{V}^*_{0S}(I) = \vec{V}^*_{0S}(A) + \vec{\Omega}^*_{0S} \times \overrightarrow{AI} = (\frac{a}{\cos\theta_0}\dot{\psi}^* + a\,\dot{\varphi}^*)\vec{n}$$

Consequently, the VP [7.89] in a VV compatible with the joint is written as

$$\mathscr{P}^*(\mathcal{F}_{0\leftrightarrow S},t) = \left[\vec{C}_A.\vec{z}_0 + \frac{a}{\cos\theta_0}\vec{F}_I.\vec{n}\right]\dot{\psi}^* + \left[\vec{C}_A.\vec{z}_S + a\vec{F}_I.\vec{n}\right]\dot{\varphi}^*$$

Definition [7.79] gives:

$$\begin{aligned}\text{The compound joint is perfect} &\Leftrightarrow \mathscr{P}^* = 0 \text{ for any compatible VV i.e. } \forall\dot{\psi}^*, \forall\dot{\varphi}^*\\ &\Leftrightarrow \boxed{\begin{array}{l}\cos\theta_0\,\vec{C}_A.\vec{z}_0 + a\,\vec{F}_I.\vec{n} = 0\\ \text{and}\quad \vec{C}_A.\vec{z}_S + a\,\vec{F}_I.\vec{n} = 0\end{array}} \qquad [7.90]\end{aligned}$$

7.6.2. *Superimposition of two perfect elementary joints*

We will now study the joint, not by considering it as a whole, as we did previously, but by considering it to be the superimposition of two elementary joints. More precisely:

(i) we will first study the two elementary joints separately, by releasing the contact at I and also the spherical joint at A. By assuming that each joint is perfect, we will derive from this the conditions on the constraint efforts exerted on the rigid body,

(ii) the compound joint is then considered to be perfect if the two elementary joints that make it up are perfect.

We will see that this procedure leads to stronger conditions than those obtained in the previous section.

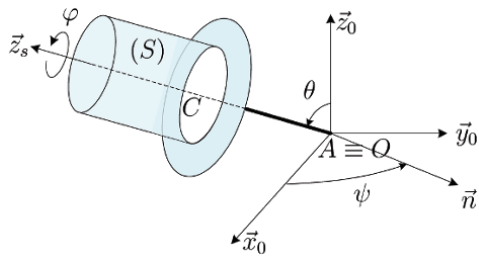


Figure 7.13. *Only the spherical joint at A*

7.6.2.1. *Perfect spherical joint at A*

As said in section 7.5.1, to study the perfection of the spherical joint at A, we must release the contact at I, i.e. remove the constraint equation [7.88] in the above parameterization. Let us thus break off the contact at I and assume that the rigid body is subject to the spherical joint only (Figure 7.13).

We will carry out the analysis using the following parameterization:

PARAMETERIZATION.

- Primitive parameters: $x, y, z, \psi, \theta, \varphi$.
- Primitive constraint equations: these are those expressing the spherical joint, namely $x = y = z = 0$.
- Retained parameters: ψ, θ, φ.
- No complementary constraint equation.

The VP of the constraint efforts exerted by the support on the rigid body S can be calculated using [5.5] and [5.8]:

$$\mathscr{P}^*(\mathcal{F}_{0\to S}, t) = \vec{F}_A.\vec{V}^*_{0S}(A) + \vec{C}_A.\vec{\Omega}^*_{0S} \qquad [7.91]$$

The VVs compatible with the joint are given by:

$$\vec{\Omega}^*_{0S} = \dot{\psi}^*\vec{z}_0 + \dot{\theta}^*\vec{n} + \dot{\varphi}^*\vec{z}_S \qquad \vec{V}^*_{0S}(A) = \vec{0}$$

Hence, the VP [7.91] in a VV compatible with the ball joint:

$$\mathscr{P}^*(\mathcal{F}_{0\leftrightarrow S}, t) = \vec{C}_A.\left(\dot{\psi}^*\vec{z}_0 + \dot{\theta}^*\vec{n} + \dot{\varphi}^*\vec{z}_S\right)$$

Definition [7.79] thus gives:

$$\text{The ball joint at } A \text{ is perfect } \Leftrightarrow \mathscr{P}^* = 0 \text{ for any compatible VV, i.e. for any } \dot{\psi}^*, \dot{\theta}^*, \dot{\varphi}^*$$

$$\Leftrightarrow \boxed{\vec{C}_A = \vec{0}} : \text{the torque at the spherical joint is zero} \qquad [7.92]$$

7.6.2.2. *Perfect point contact at I*

This time, let us free the spherical joint at A and only retain the point contact at I (Figure 7.14).

The constraint equations $x = y = z = 0$ disappear. We choose the following parameterization:

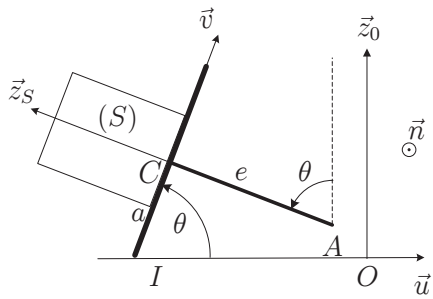


Figure 7.14. *Only the point contact at I*

PARAMETERIZATION.

- Primitive parameters: $x, y, z, \psi, \theta, \varphi$.
- No primitive constraint equation.
- Retained parameters: $x, y, z, \psi, \theta, \varphi$.
- Complementary constraint equation: it is the equation expressing the contact at I. Relationship [7.87] is still valid. Projecting it onto $\vec{z}_0$ this time gives:

$$z = a \sin\theta - e\cos\theta \qquad [7.93]$$

which is different from [7.88] obtained in the case of the compound joint.

The VP of the constraint efforts between the support and the rigid body S is calculated by [5.5] and [5.8]:

$$\mathscr{P}^*(\mathcal{F}_{0\to S}, t) = \vec{F}_I.\vec{V}^*_{0S}(I) \qquad [7.94]$$

The VV $\vec{V}^*_{0S}(I)$ compatible with the joint is obtained from $\vec{V}^*_{0S}(I) = \vec{V}^*_{0S}(A) + \vec{\Omega}^*_{0S} \times \overrightarrow{AI}$, where

– on the one hand,

$$\begin{aligned}\vec{V}^*_{0S}(A) &= \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0 + \dot{z}^*\vec{z}_0 \quad \text{with } \dot{z}^* = (e\sin\theta + a\cos\theta)\,\dot{\theta}^* \text{ according to [7.93]}\\ &= \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0 + (e\sin\theta + a\cos\theta)\,\dot{\theta}^*\vec{z}_0\end{aligned}$$

– on the other hand, $\vec{\Omega}^*_{0S} = \dot{\psi}^*\vec{z}_0 + \dot{\theta}^*\vec{n} + \dot{\varphi}^*\vec{z}_S$,

– as concerns $\overrightarrow{AI}$, one only has to write it in the form $\overrightarrow{AI} = e\vec{z}_S - a\vec{v}$ without further detail.

This gives

$$\vec{V}^*_{0S}(I) = \dot{x}^*\vec{x}_0 + \dot{y}^*\vec{y}_0 + \dot{\psi}^*(e\sin\theta + a\cos\theta)\vec{n} + \dot{\theta}^*\left[(e\sin\theta + a\cos\theta)\vec{z}_0 - e\vec{v} - a\vec{z}_S\right] + a\dot{\varphi}^*\vec{n}$$

By inserting this expression in the VP [7.94] and by applying definition [7.79], we find:

$$\text{The contact joint at } I \text{ is perfect} \;\Leftrightarrow\; \mathscr{P}^* = 0 \quad \text{for any compatible VV, i.e. } \forall \dot{x}^*, \dot{y}^*, \dot{\psi}^*, \dot{\theta}^*, \dot{\varphi}^*$$

$$\Leftrightarrow \begin{cases} \vec{F}_I.\vec{x}_0 = \vec{F}_I.\vec{y}_0 = 0 \\ \text{and } \vec{F}_I.\vec{n} = 0 \\ \text{and } \vec{F}_I.\left[(e\sin\theta + a\cos\theta)\vec{z}_0 - e\vec{v} - a\vec{z}_S\right] = 0 \end{cases}$$

Knowing that $\vec{v} = \cos\theta\vec{u} + \sin\theta\vec{z}_0$ and $\vec{z}_S = -\sin\theta\vec{u} + \cos\theta\vec{z}_0$, it can be easily verified that we arrive at the following equivalence:

$$\text{The contact at } I \text{ is perfect } \Leftrightarrow \boxed{\vec{F}_I = F_I\,\vec{z}_0}: \text{ the contact is frictionless} \qquad [7.95]$$

7.6.2.3. *Conclusion*

It can be observed that the union of the conditions for perfection [7.92] and [7.95] is *stronger* than conditions [7.90]: conditions [7.92] and [7.95] imply [7.90], but *not conversely*. In other words:

– if the two elementary joints (spherical joint and point contact) are perfect, then the combined joint resulting from the juxtaposition of these two joints is also perfect,

– however, the converse is not true: contrary to what we may believe, the compound joint may be perfect without its component joints being perfect.

REMARK. The study of the perfection of the joints being achieved, let us now see how we can solve the problem using Lagrange's equations. We can specify two cases:

1. Let us consider that the two joints – spherical joint and point contact – are perfect. We thus have conditions [7.92] and [7.95]. To write Lagrange's equations, we choose the following parameterization:

   – Primitive parameters: $x, y, z, \psi, \theta, \varphi$.

   – Primitive constraint equations: these are those expressing the spherical joint, namely $x = y = z = 0$.

   – Retained parameters: ψ, θ, φ.

   – Complementary constraint equation: the equation expressing the contact at I, namely $\theta = \theta_0 \equiv \arctan\dfrac{e}{a}$.

   We obtain seven equations in total – six Lagrange's equations and the complementary constraint equation – for seven unknowns, namely three kinematic unknowns ψ, θ, φ and four unknown constraint efforts $\vec{F}_A, F_I$.

2. Let us now assume that the compound joint is perfect, without the elementary joints being perfect. This time we have conditions [7.90]. To write Lagrange's equations, let us choose the same parameterizations as above.

   We obtain nine equations in total – six Lagrange's equations, the complementary constraint equation and the two relationships [7.90] – for 12 unknowns, namely three kinematic unknowns ψ, θ, φ and nine unknown constraint efforts $\vec{F}_A, \vec{C}_A, \vec{F}_I$. Thus, the condition for the perfection of the compound joint does not provide sufficient information and we are faced with a hyperstatic system to the third degree. ■

8

Lagrange's Equations in the Case of Perfect Joints

Consider a system $\mathcal{S}$ composed of one or more rigid bodies. The mechanical joints existing in this system are expressed by a certain number of constraint equations, which may be classified in different manners, either as primitive equations or as complementary equations. The retained parameters are $q \equiv (q_1, \ldots, q_n)$ and t.

It is assumed that we know a *Galilean* reference frame R_g. Let us recall that the Lagrange's equations are $\forall i \in [1, n], C_i = Q_i = D_i + L_i$, where D_i (respectively, L_i) is the ith generalized force corresponding to the given efforts (respectively, constraint efforts).

By decomposing the given efforts into those that are derivable from a potential $\mathcal{V}_{R_g}$ and those that are not derivable from a potential, we have $D_i = -\frac{\partial \mathcal{V}_{R_g}}{\partial q_i} + D'_i$, and, as a result, the Lagrange's equations can be written as

$$\forall t, \forall i \in [1, n], \qquad \frac{d}{dt}\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i} + \frac{\partial \mathcal{V}_{R_g}}{\partial q_i} = D'_i + L_i \qquad [8.1]$$

Unlike the D'_i coefficients, the L_i coefficients are more tedious to calculate. As said in [6.6], in the general case, we must first analyze the existing mechanical joints and identify the constraint efforts $\mathcal{F}_{\text{constraint}\to\mathcal{S}}$ applied on the system; we must then calculate the VP of the constraint efforts and, finally, from this we derive the L_i coefficients.

In this chapter, it will be seen that the hypothesis of *perfect* joints enables one to obtain expressions for the L_i coefficients systematically, easily and quickly.

• As with the Lagrange's equations in the general case (Chapter 6), we will adopt convention [6.1], according to which, the Galilean reference frame R_g being known, we choose the common reference frame R_0 equal to R_g:

$$\boxed{R_0 = R_g}$$

As was said at the beginning of Chapter 6, this choice simplifies the discussion later on as the rotation tensor of R_g with respect to R_0 is then equal to the identity tensor, $\bar{\bar{Q}}_{0g} = \bar{\bar{I}}$, in consequence of which the pair (R_g, R_0) *automatically satisfies hypothesis* [2.33] on perfect joints, which appears repeatedly, especially in Chapter 7:

HYPOTHESIS [2.33]: The rotation tensor $\bar{\bar{Q}}_{0g}$ and the point O_g fixed in R_g do *not* depend on q.

This hypothesis implies that the VV of a particle and the VP of an efforts system are *independent of the reference frame* R_g and that they can, thus, be written *without the reference frame index*:

$$\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i^* \qquad \text{and} \qquad \mathscr{P}^*(\mathcal{F}_{\to\mathcal{S}}, t)$$

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

Hypothesis [2.33] is essential to establish Lagrange's equations in the presence of perfect joints for the simple reason that the concept of the perfect joint itself was introduced in Chapter 7 under hypothesis [2.33].

Recall that this hypothesis implies the independence of the VP of constraint efforts with respect to the reference frame and that, in turn, this independence makes the concept of a perfect joint an intrinsic one. The weaker hypothesis, hypothesis [2.26], would not have been sufficient.

8.1. Lagrange's equations in the case of perfect joints and an independent parameterization

It is assumed here that the chosen parameterization is *independent*, i.e. that *all constraint equations are primitive*.

8.1.1. *Lagrange's equations*

Theorem

HYPOTHESES:

(i) The reference frame R_g is Galilean and we choose $R_0 = R_g$, as per convention [6.1].

(ii) All the joints are *perfect*.

(iii) There is *no* complementary constraint equation (i.e. the parameterization is *independent*).

The Lagrange's equations then write as

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{d}{dt}\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i} + \frac{\partial \mathcal{V}_{R_g}}{\partial q_i} = D'_i} \qquad [8.2]$$

PROOF. Consider an arbitrary n-tuple $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$. According to definition [7.2], the associated VVF, $\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i^*$, is compatible with all joints since there are no complementary equations expressing the joints. According to definition [7.79] of a perfect joint, the VP of the constraint efforts in this VVF is zero. All this can be rendered in mathematical terms as follows:

$$\forall t, \forall (\dot{q}_i^*)_{1\leq i\leq n}, \qquad \mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n} L_i \dot{q}_i^* = 0$$

Hence, $L_i = 0, \forall i \in [1, n]$, and the Lagrange's equation [8.1] gives [8.2]. ■

8.1.2. *Review*

Since the generalized forces D'_i for given efforts that are not derivable from a potential are known, the Lagrange's equations [8.2] make up a system of n second-order differential equations in time, for n unknown *functions* $(q_i)_{1\leq i\leq n}$.

In connection with section 6.3 on the need of modeling joints, we see that *the hypothesis of perfect joints allows us to systematically obtain as many equations as unknowns*, and that the equations obtained form the equations of motion for the mechanical system. We will see that we still have as many equations as unknowns in the presence of complementary constraint equations.

8.1.3. *Particular case*

The following particular case can immediately be derived from theorem [8.2]:

Definition and corollary.

HYPOTHESIS: In addition to the hypothesis in theorem [8.2], it is assumed that all the given efforts admit a potential $\mathcal{V}_{R_g}(q,t)$.

By defining the *Lagrangian of the system* $\mathcal{S}$ as

$$\boxed{\mathbb{L}(q,\dot{q},t) \equiv E^c_{R_g\mathcal{S}}(q,\dot{q},t) - \mathcal{V}_{R_g}(q,t)}$$

we can write the Lagrangian equations as

$$\forall i \in [1,n], \qquad \boxed{\frac{d}{dt}\frac{\partial \mathbb{L}}{\partial \dot{q}_i} - \frac{\partial \mathbb{L}}{\partial q_i} = 0} \qquad [8.3]$$

which shows that, under the adopted hypothesis, the Lagrangian $\mathbb{L}$ can alone determine the equations of motion. We say that the system $\mathcal{S}$ under consideration is a *Lagrangian system*.

When the constraints are independent of time, $E^c_{R_g\mathcal{S}} + \mathcal{V}_{R_g}$ is the mechanical energy of the system. We must be careful not to confuse $E^c_{R_g\mathcal{S}} + \mathcal{V}_{R_g}$ and the Lagrangian $E^c_{R_g\mathcal{S}} - \mathcal{V}_{R_g}$.

REMARK. We can relate the previous corollary and the well-known Hamilton principle in mechanics:

Hamilton's Principle. Let $\mathcal{S}$ be a mechanical system defined by n parameters $q \equiv (q_1,\ldots,q_n)$ and satisfying the hypotheses of this section. Let us assume that this system begins with a given position at an instant t_0 to arrive at another given position at t_1. The trajectory described by the system in the q space, between the two instants t_0 and t_1, is the trajectory that makes the integral $\int_{t_0}^{t_1} \mathbb{L}(q,\dot{q},t)dt$ – called *the action integral – stationary*.

In other words, of all possible motions between t_0 and t_1, the actual motion is that which makes the integral $\int_{t_0}^{t_1} \mathbb{L}(q,\dot{q},t)dt$ *stationary*.

Hamilton's principle transforms the determination of the motion of a mechanical system into a variational calculus, whose solutions $(q_1(t),\ldots,q_n(t))$ necessarily satisfy the so-called *Euler's equations*. It can be verified here that Euler's equations are equal to Lagrange's equations [8.3]. Hamilton's principle is, in the specific case in this section, to Lagrange's equations what, in optics, Fermat's principle is to Descartes' law $\sin i = n \sin r$. ■

8.2. Lagrange's equations in the case of perfect joints and in the presence of complementary constraint equations

Let us now study the case where all the joints are perfect and where, unlike in section 8.1, it is assumed here that *the chosen parameterization contains ℓ ($\ell < n$) complementary constraint equations*, which can always be written in the differential form:

$$\boxed{\sum_{i=1}^{n} \alpha_{hi}(q,t)\,\dot{q}_i + \beta_h(q,t) = 0} \qquad h \in [1,\ell] \qquad [8.4]$$

Moreover, it is assumed that the ℓ previous relationships are linearly independent, i.e. rank $[\alpha] = \ell$, where $[\alpha]$ denotes the $\ell \times n$ matrix whose (h,i)-component is α_{hi}.

We will show that there exists in this case a simple means of calculating the generalized forces associated with the constraint efforts by means of the so-called *Lagrange multipliers*.

8.2.1. *Lagrange's equations with multipliers*

The general form [8.1] of Lagrange's equations always remains valid, where the generalized forces L_i corresponding to the constraint efforts may be determined by calculating the VP of the constraint efforts.

Here, instead, we will use the hypothesis of perfect joints in order to obtain a *systematic* expression for the L_i. This is the so-called *method of Lagrange multipliers*.

Theorem

HYPOTHESES:

(i) The reference frame R_g is Galilean and we choose $R_0 = R_g$, as per convention [6.1].

(ii) All the joints are *perfect*.

(iii) There exist ℓ ($\ell < n$) *complementary* constraint equations, written in the differential form [8.4].

The generalized constraint forces L_i are then given by

$$\forall i \in [1,n], \quad \boxed{L_i = \sum_{h=1}^{\ell} \lambda_h(t)\,\alpha_{hi}(q,t)} \qquad [8.5]$$

where the parameters q in the terms $\alpha_{hi}(q,t)$ must satisfy the complementary constraint equations and the scalars $\lambda_h(t)$, $h \in [1,\ell]$, are unknown, *a priori* time dependent, called *the Lagrange multipliers*.

Consequently, the Lagrangian equations are written as

$$\forall t, \forall i \in [1,n], \qquad \boxed{\frac{d}{dt}\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i} + \frac{\partial \mathcal{V}_{R_g}}{\partial q_i} = D'_i + \underbrace{\sum_{h=1}^{\ell} \lambda_h\,\alpha_{hi}}_{L_i}} \qquad [8.6]$$

PROOF.

(i) Let us return to the expression [6.3] for the PVP which led to the general Lagrangian equations [8.1]. It is written as, with the decompositions [5.19] and [6.4] of the generalized forces, $Q_i = D_i + L_i = -\dfrac{\partial \mathcal{V}_{R_g}}{\partial q_i} + D'_i + L_i$, $i \in [1,n]$:

$$\forall t, \forall (\dot{q}_i^*)_{1\leq i\leq n}, \qquad \sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i} + \frac{\partial \mathcal{V}_{R_g}}{\partial q_i} - D'_i\right)\dot{q}_i^*$$
$$= \mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n} L_i\dot{q}_i^* \qquad [8.7]$$

As the n-tuple $(\dot{q}_i^*)_{1\leq i\leq n}$ is arbitrary, the previous relationship gives the general Lagrange equations [8.1]. Here, we will make use of the perfection of the joints in order to specify the expression for the generalized constraint forces L_i. The difference from the proof of theorem [8.2] is as follows: due to the presence of complementary constraint equations, the VVF associated with the parameterization, namely $\vec{V}^*(p) = \sum_{i=1}^{n}\frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i^*$ with an arbitrary $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$, is not, *a priori*, compatible with the joints, and consequently, even if the joints are perfect the VP of the constraint efforts is *not* zero.

(ii) To specify the expression for the L_i, let us restrict ourselves to a VVF $\vec{V}^*(p) = \sum_{i=1}^{n}\frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i^*$ such that the n-tuple $(\dot{q}_1^*, \ldots, \dot{q}_n^*)$ satisfies $\sum_{i=1}^{n}\alpha_{hi}\dot{q}_i^* = 0, \forall h \in [1, \ell]$ and such that q in $\frac{\overrightarrow{\partial P}}{\partial q_i}(q,t)$ and $\alpha_{hi}(q,t)$ satisfies the complementary constraint equations [8.4].

According to definition [7.2], the VVF $\vec{V}^*(p) = \sum_{i=1}^{n}\frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i^*$ is compatible with all the joints and thus, by virtue of definition [7.79], the VP of the constraint efforts in this VVF is zero. Put in mathematical terms: $\forall (\dot{q}_i^*)_{1\leq i\leq n}$,

$$\sum_{i=1}^{n}\alpha_{hi}\dot{q}_i^* = 0,\ \forall h \in [1,\ell] \quad \Rightarrow \quad \mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n} L_i\dot{q}_i^* = 0 \qquad [8.8]$$

By introducing three vectors of $\mathbb{R}^n$: $\dot{q}^* \equiv \begin{pmatrix} q_1^* \\ \vdots \\ q_n^* \end{pmatrix}$, $L \equiv \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}$ and $\alpha_h \equiv \begin{pmatrix} \alpha_{h1} \\ \vdots \\ \alpha_{hn} \end{pmatrix}$ (the components of α_h are those of the hth *row* in the matrix $[\alpha]$), [8.8] can be rewritten in a more compact form, denoting the scalar product in $\mathbb{R}^n$ by a point : $\forall \dot{q}^*$,

$$\alpha_h.\dot{q}^* = 0, \quad \forall h \in [1,\ell] \qquad \Rightarrow \qquad \mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = 0, L.\dot{q}^* = 0$$

Any vector $\dot{q}^*$ orthogonal to the ℓ vectors α_h is orthogonal to vector L. From the Rouché-Fontenet theorem in mathematics, it follows that vector L belongs to the vector subspace of $\mathbb{R}^n$ spanned by the vectors α_h:

$$\exists\,(\lambda_1, \ldots, \lambda_n), \quad L = \sum_{h=1}^{\ell}\lambda_h\alpha_h : \qquad \text{this is [8.5]}$$

(iii) Inserting [8.5] in [8.7] gives

$$\forall t, \forall (\dot{q}_i^*)_{1\leq i\leq n}, \quad \sum_{i=1}^{n}\left(\frac{d}{dt}\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i} - \frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i} + \frac{\partial \mathcal{V}_{R_g}}{\partial q_i} - D'_i\right)\dot{q}_i^*$$
$$= \mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n}\left(\sum_{h=1}^{\ell}\lambda_h\alpha_{hi}\right)\dot{q}_i^* \qquad [8.9]$$

Because the n-tuple $(\dot{q}_i^*)_{1\leq i\leq n}$ is arbitrary, the previous relationship gives the Lagrange equations [8.6]. ■

We must take into account the complementary constraint equations to calculate the left-hand side as well as the right-hand side of Lagrange's equations [8.6], yet not at the same time:

– For the left-hand side: when we differentiate $E^c_{R_g\mathcal{S}}(q,\dot{q},t)$ and $\mathcal{V}_{R_g}$, we must consider that the $q_i, \dot{q}_i$ are independent and we can only use the complementary constraint equations *after* having obtained the derivatives $\frac{\partial E^c_{R_g\mathcal{S}}}{\partial \dot{q}_i}(q,\dot{q},t)$, $\frac{\partial E^c_{R_g\mathcal{S}}}{\partial q_i}(q,\dot{q},t)$ and $\frac{\partial V}{\partial q_i}(q,t)$. This is the rule stated in section 6.2.

– For the right-hand side: the generalized constraint forces L_i are linear combinations of $\alpha_{hi}(q,t)$, and the parameters q that appear in the $\alpha_{hi}(q,t)$ must satisfy all the existing *complementary* constraint equations.

  This comes directly from the very proof of expression [8.5] for the L_i, where we have used the VVFs compatible with the joints. Moreover, definition [7.2] of a compatible VVF requires operation [7.6].

Having said this, the slight difference in time goes completely unnoticed in practice.

Finally, let us note that if we replace L_i in [8.5] with its definition $L_i \equiv \int_{\mathcal{S}} \vec{f}_{\text{constraint}}\cdot\frac{\overrightarrow{\partial P}}{\partial q_i}dm$, where $\vec{f}_{\text{constraint}}$ denotes the constraint forces, we have

$$\int_{\mathcal{S}} \vec{f}_{\text{constraint}}\cdot\frac{\overrightarrow{\partial P}}{\partial q_i}dm = \sum_{h=1}^{\ell}\lambda_h(t)\,\alpha_{hi}(q,t)$$

The parameters q in $\frac{\overrightarrow{\partial P}}{\partial q_i}$ on the left-hand side and those in $\alpha_{hi}(q,t)$ on the right-hand side all satisfy the same complementary constraint equations, which is consistent.

8.2.2. *Practical calculation using Lagrange's multipliers*

There is no use in learning by heart expression $L_i = \sum_{h=1}^{\ell}\lambda_h\alpha_{hi}$ in [8.5]–[8.6]. Instead, note that the VP $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}})$ is written according to [8.5]:

$$\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n}L_i\dot{q}_i^* = \sum_{i=1}^{n}\sum_{h=1}^{\ell}\lambda_h\alpha_{hi}\dot{q}_i^* = \sum_{h=1}^{\ell}\lambda_h\left(\sum_{i=1}^{n}\alpha_{hi}\dot{q}_i^*\right) \qquad [8.10]$$

From this, we can derive the following practical procedure to construct the right-hand side L *directly from the complementary constraint equations*:

1. First of all, we "virtualize" the complementary constraint equations [8.4]:

$$\sum_{i=1}^{n} \alpha_{hi}\dot{q}_i + \beta_h = 0, \quad h \in [1,\ell] \qquad \Rightarrow \qquad \sum_{i=1}^{n} \alpha_{hi}\dot{q}_i^*, \quad h \in [1,\ell]$$

2. We then write the VP of the constraint efforts as a linear combination of the virtualized forms, with the coefficients of the combination being the Lagrange's multipliers λ_h:

$$\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{h=1}^{\ell} \lambda_h \left(\sum_{i=1}^{n} \alpha_{hi}\dot{q}_i^* \right)$$

3. Finally, we derive the right-hand side L_i by identifying the previous expression with

$$\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n} L_i\dot{q}_i^*$$

EXAMPLE. Consider a disc S, with center C and radius R, homogeneous and of mass m, rolling without slipping over the axis $O\vec{x}_0$ of the Galilean reference frame $R_g = R_0$ (see Figure 7.11). The disc is subjected to the gravity field $-g\vec{y}_0$ and to a constant torque $\Gamma\vec{z}_0$. The *a priori* position of the disc in R_0 is defined by the coordinates (x, y) of center C and the rotation angle φ of the disc. We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: x, y, φ.
- No primitive constraint equation.
- Retained parameters: x, y, φ.
- Complementary constraint equations: $y = R$ expressing the contact with the axis $O\vec{x}_0$, and $\dot{x} + R\dot{\varphi} = 0$, which is the no-slip condition.

We have seen in section 7.5.5 that the no-slip contact is a perfect joint. To obtain the expressions for the L_i coefficients in [8.6], we virtualize the complementary constraint equations and then weight the relationships obtained using the λ_h:

$$\begin{cases} y = R & \Rightarrow \dot{y}^* & \Rightarrow \lambda_1\dot{y}^* \\ \dot{x} + R\dot{\varphi} = 0 & \Rightarrow \dot{x}^* + R\dot{\varphi}^* & \Rightarrow \lambda_2(\dot{x}^* + R\dot{\varphi}^*) \end{cases}$$

The VP of the constraint efforts is a linear combination of the virtualized forms: $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to S}) = T\lambda_1\dot{y}^* + \lambda_2(\dot{x}^* + R\dot{\varphi}^*)$.

By identifying this expression with $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to S}) = \sum_{i=1}^{n} L_i\dot{q}_i^*$, we derive

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \begin{pmatrix} \lambda_2 \\ \lambda_1 \\ \lambda_2 R \end{pmatrix}$$

Knowing that the parameterized kinetic energy is $E_{0S}^c = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\varphi}^2$ ($I = \frac{1}{2}mR^2$) and the potential is $\mathcal{V}_0 = mgy + const$, Lagrange's equations [8.6] can be

written as

$$\begin{array}{lll} \mathscr{L}_x : & m\ddot{x} & = \lambda_2 \\ \mathscr{L}_y : & m\ddot{y} & = \lambda_1 - mg \\ \mathscr{L}_\varphi : & I\ddot{\varphi} & = \Gamma + \lambda_2 R \end{array} \qquad \blacksquare$$

By comparing the previous Lagrange equations with those found in section 6.7 using the general method, it can be observed that $\lambda_1 = N$ and $\lambda_2 = T$, i.e. the Lagrange multipliers are equal to the constraint efforts at the contact point between the disc and the axis $O\vec{x}_0$.

In general, we can draw out *the mechanical significance of the multipliers* λ_h by directly calculating the VP of the constraint efforts and then by identifying this with expression [8.10], $\mathscr{P}^*(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n}\sum_{h=1}^{\ell} \lambda_h \alpha_{hi} \dot{q}_i^*$. That being said, the above-described operation is rather useless inasmuch as the multipliers were introduced precisely to avoid actually calculating the VP of constraint efforts.

8.2.3. *Review*

In connection with section 6.3, on the need of modeling the joints, it can be seen that, as in the case of the absence of complementary constraint equations, *the hypothesis of perfect joints enables us to obtain as many equations as unknowns*. The overview of equations and unknowns is as follows:

– on the one hand, we have $n + \ell$ unknowns: $\begin{cases} q_1, \ldots, q_n : n \text{ unknowns} \\ \lambda_1, \ldots, \lambda_\ell : \ell \text{ unknowns} \end{cases}$

– on the other hand, we have $n + \ell$ equations:

$$\begin{cases} n \text{ Lagrange's equations [8.6]} \\ \ell \text{ complementary constraint equations [8.4]: } \displaystyle\sum_{i=1}^{n} \alpha_{hi}\dot{q}_i + \beta_h = 0, \forall h \in [1, \ell]. \end{cases}$$

The perfection of the joints is a modeling, a law of physics that makes it possible to have as many equations as there are unknowns and to solve the mechanical problem. It is the analog of the so-called constitutive law in the mechanics of deformable media.

Instead of unknowns that are components of the constraint efforts, we have unknown multipliers, introduced through mathematical argument. Of course, the multipliers do not necessarily have any direct mechanical significance, but the multiplier method has the advantage of yielding the Lagrange's equations [8.6] *directly from the constraint equations*, without going through the actual calculation of the VP of the constraint efforts.

We can decide to eliminate the unknown multipliers λ_h from the Lagrange's equations [8.6] in order to reduce these to a system of *equations of motion* of n equations with the n unknowns $(q_1, \ldots, q_n)$. Elimination is easy as the multipliers are involved in a linear manner.

The multipliers method is particularly advantageous when one is only interested in the kinematic unknowns and not in the unknown constraint efforts.

8.2.4. *Remarks*

1. The multipliers method applies only when all the joints are *perfect*. These are all the *external or internal* joints in the studied mechanical system $\mathcal{S}$.

   Recall that the mechanical joints imposed on the system $\mathcal{S}$ are generally made up of internal joints (joints between two rigid bodies in $\mathcal{S}$) and joints with the exterior (joints between a rigid body in $\mathcal{S}$ and a rigid body external to $\mathcal{S}$).

2. In Chapter 6 we saw that Lagrange's equations contain the constraint efforts ensuring the *complementary* constraint equations, not those ensuring the primitive constraint equations. Thus, we must use the appropriate parameterization by choosing to classify such a constraint equation as complementary, depending on the constraint efforts that we wish to calculate.

   Likewise, it can be seen that Lagrange's equations [8.6] only contain information (via the multipliers) concerning the constraint efforts that ensure *complementary* constraint equations. Indeed, consider a perfect mechanical joint of which all constraint equations are written as primitive; the VVF associated with the parameterization is automatically compatible with this joint and thus, the VP of the constraint inter-efforts is zero in any VVF. Consequently, the efforts ensuring this joint do not appear in the VP [8.10].

   Given that the multipliers method provides information on the constraint efforts that ensure the *complementary constraint efforts*, if we wish to have access to this information, we must not use an independent parameterization but rather a parameterization with well-chosen complementary equations.

   We can even select among the constraint efforts and calculate the corresponding generalized forces L_i in two different ways:

   – some L_i are obtained using formula [8.5] (method of Lagrange multipliers);

   – others are obtained using an actual calculation of the VP of constraint efforts.

8.3. Example: particle on a rotating hoop

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and a system consisting of a hoop C with radius a and a particle p, whose position in R_0 is P and whose mass is m.

The position vector of p is denoted by $\overrightarrow{OP} = r\,\vec{z}$, where r is the radial distance and $\vec{z}$ is a unit vector (Figure 8.1). We define the unit vector $\vec{n}$ such that the basis $(\vec{z}_0, \vec{z}, \vec{n})$ is right-handed and the unit vector $\vec{v} \equiv \vec{z} \times \vec{n}$, which allows us to then define the angle $\theta \equiv (\vec{z}_0, \vec{z})$ measured around $\vec{n}$ and the angle $\psi \equiv (\vec{x}_0, \vec{n})$ measured around $\vec{z}_0$.

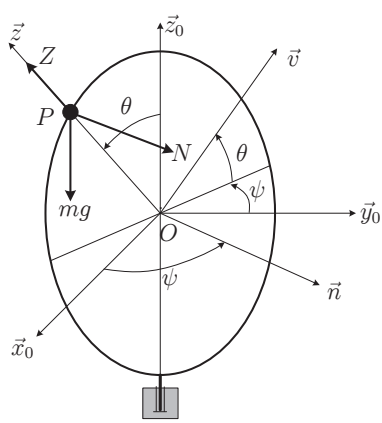


Figure 8.1. *Particle moving around a hoop*

The primitive parameters of p are r, ψ, θ.

The hoop's position is defined by the angle denoted by $\alpha \equiv (\vec{x}_0, \vec{c})$, measured around $\vec{z}_0$, where $\vec{c}$ is a unit vector orthogonal to the hoop (the axis of the hoop is thus $O\vec{c}$).

The system moving in the gravity field $-g\vec{z}_0$ is subjected to the following joints:

– The hoop (C) is pinned to axis $O\vec{z}_0$, in such a way that its center coincides with the origin O. A servomotor makes the hoop rotate at the constant angular velocity $\omega > 0$. It is assumed that $\vec{c} = \vec{x}_0$ at the initial instant $t = 0$, and thus $\alpha = \omega t$.

– The particle p is constrained to remain on the hoop. The joint between the particle and the hoop is expressed by two constraints equations: $r = a$ and $\psi = \alpha$ (i.e. $\vec{n} = \vec{c}$). It is assumed that the joint between the particle and the hoop is perfect. As seen in section 7.5.4, this amounts to assuming that there is no friction between the particle and the hoop.

We will establish the Lagrange's equations by choosing several different parameterizations and we will see that the information obtained varies depending on the parameterization chosen.

8.3.1. *Independent parameterization*

We first choose the following parameterization:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters: r, ψ, θ and α.
- Primitive constraint equations: $r = a$, $\alpha = \omega t$ and $\psi = \alpha = \omega t$.
- Retained parameters: θ, t. Hence $\overrightarrow{OP} = a\ \vec{z}(\omega t, \theta)$. The position of the hoop (C) depends only on t.
- No complementary constraint equation.

The velocity of the particle p associated with the independent parameterization is

$$\vec{V}_{R_0}(p) = \frac{\partial \overrightarrow{OP}}{\partial \theta}\dot{\theta} + \frac{\partial \overrightarrow{OP}}{\partial t} = -a\dot{\theta}\ \vec{v} + a\omega \sin\theta\ \vec{n}$$

The kinetic energy of the particle is a function of $\theta, \dot{\theta}$:

$$E^c_{R_0 p} = \frac{1}{2} m \vec{V}_0^2(p) = \frac{1}{2} m a^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right)$$

The kinetic energy of the hoop is $\frac{1}{2} I \omega^2$, where I is the moment of inertia of the hoop about the axis $O\vec{z}_0$. This energy is constant and does not come into play in the Lagrange equations, which involves the derivatives of the kinetic energy.

By denoting z the elevation of P in the coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$, the weight of the particle is derivable from the potential

$$\mathcal{V}_{R_0} = mgz + const = mga\cos\theta + const$$

The potential due to the weight of the hoop is constant and does not come into play. As the joint is perfect and the parameterization is independent, the Lagrange's equation [8.2] gives

$$a(\ddot{\theta} - \omega^2 \sin\theta\cos\theta) - g\sin\theta = 0 \qquad [8.11]$$

We obtain an equation of motion in θ, which is a nonlinear, second-order differential equation in time.

By multiplying the equation of motion by $\dot{\theta}$ and then integrating with respect to time, we arrive at a first integral of the form $A\dot{\theta}^2 = F(\theta, IC)$, where IC formally stands for the initial conditions:

$$a\dot{\theta}^2 = -\frac{a\omega^2}{2}\cos 2\theta - 2g\cos\theta + const \qquad [8.12]$$

8.3.2. *Reduced parameterization no. 1*

Let us now choose the following parameterization:

REDUCED PARAMETERIZATION NO. 1.

- Primitive parameters: r, ψ, θ and α.
- Primitive constraint equations: $r = a$ and $\alpha = \omega t$.
- Retained parameters: ψ, θ, t. Hence $\overrightarrow{OP} = a\ \vec{z}(\psi,\theta)\ \vec{z}(\psi,\theta)$. The position of the hoop (C) depends only on t.
- Complementary constraint equation: $\psi = \alpha = \omega t$.

The velocity of the particle associated with this reduced parameterization is

$$\vec{V}_{R_0}(p) = \frac{\partial\overrightarrow{OP}}{\partial\psi}\dot{\psi} + \frac{\partial\overrightarrow{OP}}{\partial\theta}\dot{\theta} + \frac{\partial\overrightarrow{OP}}{\partial t} = a\sin\theta\ \dot{\psi}\ \vec{n} - a\dot{\theta}\ \vec{v}$$

This time, the kinetic energy of the particle is, *a priori*, a function of $(\psi, \theta, \dot{\psi}, \dot{\theta})$ (actually, here there is no ψ):

$$E^c_{R_0 p} = \frac{1}{2}m\vec{V}_0^2(p) = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right)$$

The weight is derivable from the potential

$$\mathcal{V}_{R_0} = mgz + const = mga\cos\theta + const$$

The joint is still perfect, however this time as there is a complementary constraint equation, we must apply the Lagrange equation [8.6], λ denoting the Lagrange multiplier:

$$\begin{aligned} \mathscr{L}_\theta :\ & a(\ddot{\theta} - \dot{\psi}^2\sin\theta\cos\theta) - g\sin\theta &= 0 \\ \mathscr{L}_\psi :\ & ma^2\frac{d}{dt}(\dot{\psi}\sin^2\theta) &= \lambda \end{aligned}$$

Taking into account the complementary constraint equation $\psi = \omega t$, we arrive at two equations for two unknowns θ and λ:

$$\begin{aligned} & a(\ddot{\theta} - \omega^2\sin\theta\cos\theta) - g\sin\theta &= 0 \\ & 2ma^2\omega\sin\theta\cos\theta\ \dot{\theta} &= \lambda \end{aligned} \qquad [8.13]$$

The first equation is the equation of motion [8.11] obtained above using the independent parameterization. The second equation allows one to determine, if we wish to, the Lagrange multiplier λ, once we have obtained $\theta(t)$.

• Optionally, we can find the physical significance of the multiplier λ by directly calculating the VP of the constraint efforts:

$$\mathscr{P}^*(\mathcal{F}_{C\leftrightarrow p}, t) = \vec{F}_{C\to p}.\vec{V}_C^*(p) \qquad [8.14]$$

where the constraint force exerted by the hoop (C) on the particle p is written as $\vec{F}_{C\to p} = N\vec{n} + Z\vec{z}$, and the relative VV $\vec{V}_C^*(p)$ of p with respect to (C) is calculated using the composition formula for velocities [4.43]:

$$\vec{V}_C^*(p) = \vec{V}_{R_0}^*(p) - \vec{V}_{R_0C}^*(P) \qquad [8.15]$$

where $\vec{V}_{R_0C}^*(P)$ is the VV, with respect to R_0, of the particle of the hoop C that is located, at instant t, at the position P of p (this is the background virtual velocity).

The VV $\vec{V}_{R_0}^*(p)$ associated with the parameterization under consideration is

$$\vec{V}_{R_0}^*(p) = \frac{\overrightarrow{\partial P}}{\partial \psi}\dot{\psi}^* + \frac{\overrightarrow{\partial P}}{\partial \theta}\dot{\theta}^* = a\sin\theta\ \dot{\psi}^*\ \vec{n} - a\dot{\theta}^*\ \vec{v} \qquad [8.16]$$

The background virtual velocity $\vec{V}_{R_0C}^*(P)$ is calculated using the velocity field relationship [4.35]:

$$\vec{V}_{R_0C}^*(P) = \vec{V}_{R_0C}^*(O) + \vec{\Omega}_{R_0C}^* \times \overrightarrow{OP} = \vec{0} \qquad [8.17]$$

since $\vec{V}_{R_0C}^*(O) = \vec{0}$ and $\vec{\Omega}_{R_0C}^* = \vec{0}$ (the position of the hoop (C) depends only on t). Inserting the relationships [8.16] and [8.17] into [8.15] gives the relative VV $\vec{V}_C^*(p) = -a\dot{\theta}^*\ \vec{v} + a\sin\theta\ \dot{\psi}^*\ \vec{n}$. Hence, by [8.14]:

$$\mathscr{P}^*(\mathcal{F}_{C\leftrightarrow p}, t) = (N\vec{n} + Z\vec{z}).(N\vec{n} + Z\vec{z}).(a\sin\theta\ \dot{\psi}^*\ \vec{n} - a\dot{\theta}^*\ \vec{v}) = Na\sin\theta\ \dot{\psi}^*$$

Identifying this expression with $\mathscr{P}^* = \lambda\dot{\psi}^*$ yields

$$\lambda = Na\sin\theta$$

The multiplier λ has the dimensions of a torque; its value is proportional to the normal constraint force N exerted by the hoop on to the particle, directed along vector $\vec{n}$.

In general, if we consider that the VP of constraint efforts has the dimensions of a power, we can derive from relationship [8.10] the physical dimension of the Lagrange multipliers λ_h, knowing those of the virtual parameters $\dot{q}_i^*$ and the functions α_{hi}. Thus, we often find that the multipliers are linear combinations of constraint efforts (i.e. constraint forces or torques).

8.3.3. *Reduced parameterization no. 2*

Consider the following parameterization where the constraint equation $r = a$ is put into complementary equations:

REDUCED PARAMETERIZATION NO. 2.

- Primitive parameters: r, ψ, θ and α.
- Primitive constraint equation: $\alpha = \omega t$.
- Retained parameters: r, ψ, θ, t. Hence $\overrightarrow{OP} = r\ \vec{z}(\psi, \theta)\ \vec{z}(\psi, \theta)$. The position of the hoop (C) depends only on t.
- Complementary constraint equations: $r = a$ and $\psi = \alpha = \omega t$.

The velocity of the particle associated with this reduced parameterization is

$$\vec{V}_{R_0}(p) = \frac{\partial \overrightarrow{OP}}{\partial r}\dot{r} + \frac{\partial \overrightarrow{OP}}{\partial \psi}\dot{\psi} + \frac{\partial \overrightarrow{OP}}{\partial \theta}\dot{\theta} + \frac{\partial \overrightarrow{OP}}{\partial t} = \dot{r}\ \vec{z} + r\sin\theta\ \dot{\psi}\ \vec{n} - r\dot{\theta}\ \vec{v}$$

The kinetic energy of the particle is, *a priori*, a function of $(r, \psi, \theta, \dot{r}, \dot{\psi}, \dot{\theta})$ (actually, here there is no ψ):

$$E^c_{R_0 p} = \frac{1}{2}m\vec{V}_0^2(p) = \frac{1}{2}m\left(r^2\dot{\theta}^2 + r^2\dot{\psi}^2\sin^2\theta + \dot{r}^2\right)$$

The weight is derivable from the potential

$$\mathcal{V}_{R_0} = mgz + const = mgr\cos\theta + const$$

The Lagrange's equations [8.6] give three equations by denoting the Lagrange multipliers by λ and μ:

$$\begin{array}{lll} \mathscr{L}_\theta : & \dfrac{d}{dt}(r^2\dot{\theta}) - r^2\dot{\psi}^2\sin\theta\cos\theta - gr\sin\theta & = 0 \\ \mathscr{L}_\psi : & m\dfrac{d}{dt}(r^2\dot{\psi}\sin^2\theta) & = \lambda \\ \mathscr{L}_r : & m\left[\ddot{r} - r(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta)\right] + mg\cos\theta & = \mu \end{array}$$

Taking into account the complementary constraint equations $r = a$, $\psi = \omega t$, we arrive at three equations for three unknowns θ, λ and μ:

$$\begin{array}{ll} a(\ddot{\theta} - \omega^2\sin\theta\cos\theta) - g\sin\theta & = 0 \\ 2\omega m a^2\sin\theta\cos\theta\ \dot{\theta} & = \lambda \\ -ma(\dot{\theta}^+\omega^2\sin^2\theta) + mg\cos\theta & = \mu \end{array} \qquad [8.18]$$

The first equation is the same equation of motion obtained above by means of the other parameterizations. The two other equations allows us to determine, if we wish to, the Lagrange multipliers λ, μ, once $\theta(t)$ has been obtained.

Direct calculation of the VP of the constraint efforts gives

$$\begin{array}{rcl} \mathscr{P}^*(\mathcal{F}_{C\leftrightarrow p}, t) & = & \vec{F}_{C\to p}.\vec{V}^*_C(p) = (N\vec{n} + Z\vec{z}).(N\vec{n} + Z\vec{z}).(\dot{r}^*\ \vec{z} + r\sin\theta\ \dot{\psi}^*\ \vec{n} - r\dot{\theta}^*\ \vec{v}) \\ & = & Z\dot{r}^* + Nr\sin\theta\ \dot{\psi}^* \end{array}$$

By identifying this expression with $\mathscr{P}^* = \lambda\dot{\psi}^* + \mu\dot{r}^*$, we arrive at

$$\lambda = Nr\sin\theta \qquad \mu = Z$$

From the last two equations in [8.18], we can derive the constraint forces

$$N = 2ma\omega\cos\theta\ \dot{\theta} \qquad Z = -ma(\dot{\theta}^2 + \omega^2\sin^2\theta) + mg\cos\theta$$

By replacing $\dot{\theta}$ with its expression from the first integral [8.12], we can express the forces N and Z as functions of $\cos\theta$ (and the initial conditions) alone. We can also obtain the maximal values of the constraint force exerted by the particle on the hoop.

8.3.4. *Calculation of the engine torque*

We now wish to calculate the engine torque exerted on the hoop and which ensures the hoop rotation at a constant angular velocity. In order to do this, let us choose the following parameterization:

REDUCED PARAMETERIZATION NO. 3.

- Primitive parameters: r, ψ, θ and α.
- Primitive constraint equations: $\psi = \alpha$ and $r = a$.
- Retained parameters: α, θ. Hence $\overrightarrow{OP} = a\,\vec{z}(\psi, \theta) = a\,\vec{z}(\alpha, \theta)$. The position of the hoop (C) is determined by α.
- Complementary constraint equation: $\alpha = \omega t$.

The kinetic energy of the hoop (C) is $E^c_{0C} = \dfrac{1}{2} I \dot{\alpha}^2$. The velocity of the particle p associated with this parameterization is

$$\vec{V}_{R_0}(p) = \frac{\partial \overrightarrow{OP}}{\partial \theta}\dot{\theta} + \frac{\partial \overrightarrow{OP}}{\partial \alpha}\dot{\alpha} + \frac{\partial \overrightarrow{OP}}{\partial t} = -a\dot{\theta}\,\vec{v} + a \sin\theta\,\dot{\alpha}\,\vec{n}$$

Hence the kinetic energy of the particle is

$$E^c_{0p} = \frac{1}{2} m \vec{V}^2(p) = \frac{1}{2} m a^2 \left(\dot{\theta}^2 + \dot{\alpha}^2 \sin^2\theta\right)$$

The kinetic energy of the system $\mathcal{S} \equiv p \cup C$ is, therefore, $E^c_{R_0\mathcal{S}} = \dfrac{1}{2} I \dot{\alpha}^2 + \dfrac{1}{2} m a^2 \left(\dot{\theta}^2 + \dot{\alpha}^2 \sin^2\theta\right)$.

The weight is derivable from the potential

$$\mathcal{V}_{R_0} = mgz + const = mga\cos\theta + const$$

The VP of the engine torque Γ is $\mathscr{P}^*_{\text{motor}\to C} = \Gamma\dot{\alpha}^*$.

Let us now study the VP of the inter-efforts between the hoop and the particle. By denoting the force exerted by the hoop on the particle by $\vec{F}_{C\to p} \equiv N\vec{n} + Z\vec{z}$ and by denoting the opposing force exerted by the particle on the hoop by $\vec{F}_{p\to C}$, the VP of the inter-efforts can be written as

$$\mathscr{P}^*(\mathcal{F}_{C\leftrightarrow p}, t) = \vec{F}_{C\to p}.\vec{V}^*_{R_0}(p) + \vec{F}_{p\to C}.\vec{V}^*_{R_0C}(P) = \vec{F}_{C\to p}.\left(\vec{V}^*_{R_0}(p) - \vec{V}^*_{R_0C}(P)\right) \qquad [8.19]$$

This is another way of obtaining relationship [8.15].

In accordance with definition [7.2], the VV $\vec{V}^*_{R_0}(p)$ of the particle as well as the VVF $\vec{V}^*_{R_0C}(P)$ of the hoop are automatically *compatible* with the joint between the hoop and the particle, since there is no complementary constraint equation that expresses the joint (the complementary constraint equation $\alpha = \omega t$ does not concern the joint in question!). In addition, as the joint is assumed to be perfect, the VP in these VVFs of the inter-efforts between the hoop and the particle is *zero*.

REMARK. Let us verify this statement through a direct calculation. The VV of the particle is

$$\vec{V}^*_{R_0}(P) = \frac{\partial \overrightarrow{OP}}{\partial \theta}\dot{\theta}^* + \frac{\partial \overrightarrow{OP}}{\partial \alpha}\dot{\alpha}^* = -a\dot{\theta}^*\,\vec{v}(\alpha, \theta) + a\sin\theta\,\dot{\alpha}^*\,\vec{n}(\alpha)$$

On the other hand, the VV $\vec{V}^*_{R_0C}(P)$ can be obtained through

$$\vec{V}^*_{R_0C}(P) = \vec{V}^*_{R_0C}(O) + \vec{\Omega}^*_{R_0C} \times \overrightarrow{OP} = \vec{0} + \dot{\alpha}^*\vec{z}_0 \times a\vec{z} = a\sin\theta\,\dot{\alpha}^*\,\vec{n}(\alpha)$$

Hence, the VP of the inter-efforts [8.19] gives

$$\mathscr{P}^*(\mathcal{F}_{C\leftrightarrow p}, t) = (N\vec{n} + Z\vec{z}).(-a\dot{\theta}^*\ \vec{v}) = 0$$

which is exactly what we had predicted above. ■

Finally, the Lagrange's equations can be written as

$$\begin{aligned} \mathscr{L}_\alpha :\ & I\ddot{\alpha} + ma^2\frac{d}{dt}(\dot{\alpha}\sin^2\theta) &= \Gamma \\ \mathscr{L}_\theta :\ & a(\ddot{\theta} - \dot{\alpha}^2\sin\theta\cos\theta) - g\sin\theta &= 0 \end{aligned}$$

Now taking into account the complementary constraint equation $\alpha = \omega t$, we arrive at two equations for two unknowns, namely $\theta(t)$ and $\Gamma(t)$:

$$\begin{aligned} &\Gamma = 2ma^2\omega\sin\theta\cos\theta\,\dot{\theta} \\ &a(\ddot{\theta} - \omega^2\sin\theta\cos\theta) - g\sin\theta = 0 \end{aligned}$$

The second equation is the equation of motion. After having solved this equation to obtain θ as a function of time, we obtain the engine torque $\Gamma(t)$ exerted on the hoop using the first equation.

8.4. Example: rigid body connected to a rotating rod by a spherical joint (no. 1)

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and a system composed of a rigid rod OA and a rigid body S (Figure 8.2). The whole system is subjected to the gravity field $-g\vec{z}_0$.

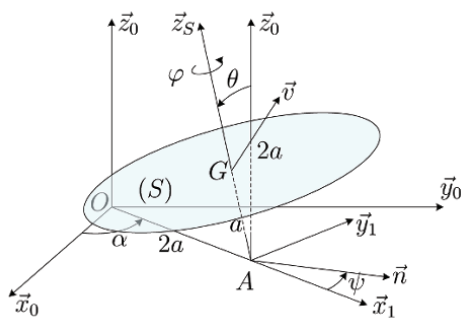


Figure 8.2. *Rigid body connected to a rotating rod by a spherical joint*

The rod OA has the length $2a$ and negligible mass. Pivoted to the vertical axis $O\vec{z}_0$, it moves in the plane $O\vec{x}_0\vec{y}_0$. We shall denote $\overrightarrow{OA} = 2a\vec{x}_1$ and $\alpha = (\vec{x}_0, \vec{x}_1)$, angle measured around $\vec{z}_0$.

The rigid body S consists of a disc with center G, radius $2a$ and negligible thickness, homogenous and of mass m; a rod with length a and negligible mass is welded onto the disc along its axis. One end of this rod is connected to the end A of the rod OA by a spherical joint. We shall denote $\overrightarrow{AG} = a\vec{z}_S$. All existing joints are perfect.

In this problem, the Euler angles (ψ, θ, φ) of the rigid body S are defined *relative to the rotating basis* $(\vec{x}_1, \vec{y}_1, \vec{z}_0)$. We thus move from the fixed basis $(\vec{x}_0, \vec{y}_0, \vec{z}_0)$ to the basis $(\vec{x}_S, \vec{y}_S, \vec{z}_S)$ attached to S through four successive rotations: α around $\vec{z}_0$, and then ψ around $\vec{z}_0$, θ around $\vec{n}$, and φ around $\vec{z}_S$, as summarized in the following sketch:

$$(\vec{x}_0, \vec{y}_0, \vec{z}_0) \overset{\alpha/\vec{z}_0}{\rightarrow} (\vec{x}_1, \vec{y}_1, \vec{z}_0) \overset{\psi/\vec{z}_0}{\rightarrow} (\vec{n}, \vec{u}, \vec{z}_0) \overset{\theta/\vec{n}}{\rightarrow} (\vec{n}, \vec{v}, \vec{z}_S) \overset{\varphi/\vec{z}_S}{\rightarrow} (\vec{x}_S, \vec{y}_S, \vec{z}_S)$$

The parameterization is as follows:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters: α, ψ, θ, φ.
- No primitive constraint equation.
- Retained parameters: α, ψ, θ, φ.
- No complementary constraint equation.

Let us calculate the kinetic energy of the system using König's formula: $E^c_{0S} = E^c_{R_0^G S} + \frac{1}{2}m\vec{V}^2_{0S}(G)$, where $E^c_{R_0^G S}$ denotes the kinetic energy calculated in the barycentric reference frame R_0^G, which has its origin in the center of mass G.

To calculate $E^c_{R_0^G S}$, on the one hand, we need the angular velocity of the rigid body S

$$\vec{\Omega}_{0S} = \dot{\theta}\,\vec{n} + (\dot{\alpha} + \dot{\psi})\sin\theta\,\vec{v} + \left[\dot{\varphi} + (\dot{\alpha} + \dot{\psi})\cos\theta\right]\vec{z}_S$$

and on the other hand, we need the inertia matrix of the disc, which can be written in any coordinate system whose origin is G and whose third vector is $\vec{z}_S$: $\mathbb{I}_S(G;\bullet,\bullet,\vec{z}_S) = ma^2\begin{bmatrix}1 & & \\ & 1 & \\ & & 2\end{bmatrix}$. We thus obtain

$$E^c_{R_0^G S} = \frac{1}{2}ma^2\left\{\dot{\theta}^2 + (\dot{\alpha} + \dot{\psi})^2\sin^2\theta + 2\left[\dot{\varphi} + (\dot{\alpha} + \dot{\psi})\cos\theta\right]^2\right\} \qquad [8.20]$$

As the velocity of the disc center is

$$\vec{V}_{0S}(G) = 2a\dot{\alpha}\,\vec{y}_1 + a(\dot{\alpha} + \dot{\psi})\sin\theta\,\vec{n} - a\dot{\theta}\,\vec{v}$$

we get

$$\vec{V}^2_{0S}(G) = 4a^2\dot{\alpha}^2 + a^2(\dot{\alpha} + \dot{\psi})^2\sin^2\theta + a^2\dot{\theta}^2 + 4a^2\dot{\alpha}(\dot{\alpha} + \dot{\psi})\sin\theta\sin\psi - 4a^2\dot{\alpha}\dot{\theta}\cos\theta\cos\psi$$

Hence the kinetic energy of the system:

$$E^c_{0S} = ma^2\left\{\dot{\theta}^2 + (\dot{\alpha} + \dot{\psi})^2\sin^2\theta + \left[\dot{\varphi} + (\dot{\alpha} + \dot{\psi})\cos\theta\right]^2 + 2\dot{\alpha}^2 + \right.$$
$$\left. 2\dot{\alpha}(\dot{\alpha} + \dot{\psi})\sin\theta\sin\psi - 2\dot{\alpha}\dot{\theta}\cos\theta\cos\psi\right\} \qquad [8.21]$$

The potential corresponding to the system's weight is

$$\mathcal{V} = mga\cos\theta + const \qquad [8.22]$$

The Lagrange's equations [8.2] give

$$\mathscr{L}_\alpha : \frac{d}{dt}\left\{2(\dot{\alpha} + \dot{\psi})sin^2\theta + 2\left[\dot{\varphi} + (\dot{\alpha} + \dot{\psi})\cos\theta\right]\cos\theta + 4\dot{\alpha} + \right.$$
$$\left. (4\dot{\alpha} + 2\dot{\psi})\sin\theta\sin\psi - 2\dot{\theta}\cos\theta\cos\psi\right\} = 0 \qquad [8.23]$$

$$\mathscr{L}_\psi : \frac{d}{dt}\left\{2(\dot\alpha+\dot\psi)sin^2\theta + 2\left[\dot\varphi + (\dot\alpha+\dot\psi)\cos\theta\right]\cos\theta + 2\dot\alpha\sin\theta\sin\psi\right\}$$
$$- 2\dot\alpha(\dot\alpha+\dot\psi)\sin\theta\cos\psi - 2\dot\alpha\dot\theta\cos\theta\sin\psi = 0 \quad [8.24]$$

$$\mathscr{L}_\theta : 2\frac{d}{dt}(\dot\theta - \dot\alpha\cos\theta\cos\psi) - 2(\dot\alpha+\dot\psi)^2\sin\theta\cos\theta + 2\sin\theta(\dot\alpha+\dot\psi)\left[\dot\varphi + (\dot\alpha+\dot\psi)\cos\theta\right]$$
$$- 2\dot\alpha(\dot\alpha+\dot\psi)cos\theta\sin\psi - 2\dot\alpha\dot\theta\sin\theta\cos\psi - \frac{g}{a}\sin\theta = 0 \quad [8.25]$$

$$\mathscr{L}_\varphi : \frac{d}{dt}\left[\dot\varphi + (\dot\alpha+\dot\psi)\cos\theta\right] = 0 \quad [8.26]$$

We have four equations [8.23]–[8.26] with the four unknowns α, ψ, θ and φ. The equations in α and φ immediately give two first integrals (these will be studied in Chapter 9) and it will be seen in section 9.5 that there exists a third first integral, which may advantageously be used to replace one of the two Lagrange equations in ψ and θ obtained above.

8.5. Example: rigid body connected to a rotating rod by a spherical joint (no. 2)

We consider again the system in the previous example, assuming this time that an engine whose stator is attached to R_0 makes the rod OA rotate at a constant angular velocity: $\dot\alpha = \omega > 0$.

8.5.1. *Total parameterization*

We choose the following parameterization where the only existing constraint equation, namely $\dot\alpha = \omega$, is written as a complementary equation:

TOTAL PARAMETERIZATION.

- Primitive parameters: α, ψ, θ, φ.
- No primitive constraint equation.
- Retained parameters: α, ψ, θ, φ.
- Complementary constraint equation: $\dot\alpha = \omega$.

The kinetic energy of the system is given by expression [8.21], the potential due to the system's weight by [8.22]. By applying [8.6], we obtain the same Lagrange equations as [8.23]–[8.26], except that here the right-hand side of the equation in α is no longer zero:

$$\mathscr{L}_\alpha : \text{ the same left-hand side as [8.23]} = \frac{\lambda}{ma^2},$$

where λ denotes the Lagrange multiplier associated with the complementary constraint equation $\dot\alpha = \omega$.

Now taking into account the complementary equation $\dot\alpha = \omega$, these Lagrange equations become:

$$\mathscr{L}_\alpha : \frac{d}{dt}\left\{2(\omega+\dot\psi)sin^2\theta + 2\left[\dot\varphi + (\omega+\dot\psi)\cos\theta\right]\cos\theta + 4\omega + \right.$$
$$\left. (4\omega + 2\dot\psi)\sin\theta\sin\psi - 2\dot\theta\cos\theta\cos\psi\right\} = \frac{\lambda}{ma^2} \quad [8.27]$$

$$\mathscr{L}_\psi : \frac{d}{dt}\left\{(\omega+\dot{\psi})sin^2\theta + \left[\dot{\varphi}+(\omega+\dot{\psi})\cos\theta\right]\cos\theta + \omega\sin\theta\sin\psi\right\} - \omega(\omega+\dot{\psi})\sin\theta\cos\psi$$
$$- \omega\dot{\theta}\cos\theta\sin\psi = 0 \quad [8.28]$$

$$\mathscr{L}_\theta : \frac{d}{dt}(\dot{\theta}-\omega\cos\theta\cos\psi) - (\omega+\dot{\psi})^2\sin\theta\cos\theta + \sin\theta(\omega+\dot{\psi})\left[\dot{\varphi}+(\omega+\dot{\psi})\cos\theta\right]$$
$$- \omega(\omega+\dot{\psi})cos\theta\sin\psi - \omega\dot{\theta}\sin\theta\cos\psi - \frac{g}{2a}\sin\theta = 0 \quad [8.29]$$

$$\mathscr{L}_\varphi : \frac{d}{dt}\left[\dot{\varphi}+(\omega+\dot{\psi})\cos\theta\right] = 0 \quad [8.30]$$

Solving the three equations [8.28]–[8.30] provides ψ, θ and φ as functions of time. Once the angles ψ, θ, φ are known, relationship [8.27] enables us to calculate the multiplier λ as a function of time.

• Let us find the mechanical significance of the multiplier λ. Let $\Gamma\vec{z}_0$ denote the torque exerted on the rod OA by the engine, which enforces $\dot{\alpha} = \omega$. Let R_1 denote the rotating reference frame defined by the coordinate system $(O; \vec{x}_1, \vec{y}_1, \vec{z}_0)$, such that the virtual angular velocity of R_1 with respect to R_0 is $\vec{\Omega}^*_{01} = \dot{\alpha}^*\vec{z}_0$. The VP of the torque $\Gamma\vec{z}_0$ is

$$\mathcal{P}^*(\Gamma\vec{z}_0) = \Gamma\vec{z}_0.\vec{\Omega}^*_{01} = \Gamma\dot{\alpha}^*$$

By identifying this expression with $\mathscr{P}^*(\Gamma\vec{z}_0) = \lambda\dot{\alpha}^*$, we arrive at $\lambda = \Gamma$: the multiplier λ is the engine torque imposing the constant rotation rate of the rod OA.

• As an exercise, let us once again find the mechanical significance of the multiplier λ using Newtonian mechanics. Let $\sigma_{0S}(O\vec{z}_0)$ and $\delta_{0S}(O\vec{z}_0)$, respectively, denote the angular momentum and the dynamic moment of the rigid body S about the axis $O\vec{z}_0$, calculated with respect to R_0 (see [1.76] for the definition of the dynamic moment).

Since the axis $O\vec{z}_0$ is fixed in R_0, we have

$$\frac{d}{dt}\sigma_{0S}(O\vec{z}_0) = \delta_{0S}(O\vec{z}_0) = \mathcal{M}_{ext\to S}(O\vec{z}_0) = \Gamma$$

where $\sigma_{0S}(O\vec{z}_0)$ is related to the angular momentum $\vec{\sigma}_{R_0^G S}(G)$ of S about G, calculated with respect to the barycentric reference frame R_0^G through

$$\sigma_{0S}(O\vec{z}_0) = \vec{\sigma}_{0S}(O)\cdot\vec{z}_0 \qquad \text{with} \qquad \vec{\sigma}_{0S}(O) = \vec{\sigma}_{R_0^G S}(G) + \overrightarrow{OG}\times m\vec{V}_{0S}(G)$$

On the one hand, using the matrix representations in the basis $(\vec{n}, \vec{v}, \vec{z}_S)$, we have

$$\vec{\sigma}_{R_0^G S}(G) = ma^2\begin{bmatrix}1 & & \\ & 1 & \\ & & 2\end{bmatrix}\begin{Bmatrix}\dot{\theta} \\ (\dot{\alpha}+\dot{\psi})\sin\theta \\ \dot{\varphi}+(\dot{\alpha}+\dot{\psi})\cos\theta\end{Bmatrix} = ma^2\begin{Bmatrix}\dot{\theta} \\ (\dot{\alpha}+\dot{\psi})\sin\theta \\ 2\left[\dot{\varphi}+(\dot{\alpha}+\dot{\psi})\cos\theta\right]\end{Bmatrix}$$

Hence:

$$\vec{\sigma}_{R_0^G S}(G).\vec{z}_0 = ma^2\left\{(\dot{\alpha}+\dot{\psi})\sin^2\theta + 2\left[\dot{\varphi}+(\dot{\alpha}+\dot{\psi})\cos\theta\right]\cos\theta\right\}$$

On the other hand:

$$(\overrightarrow{OG}\times m\vec{V}_{0S}(G)).\vec{z}_0 = ma^2\left\{4\dot{\alpha} + 2(\dot{\alpha}+\dot{\psi})\sin\theta\sin\psi - 2\dot{\theta}\cos\theta\cos\psi\right.$$
$$\left. +2\dot{\alpha}\sin\theta\sin\psi + (\dot{\alpha}+\dot{\psi})\sin^2\theta\right\}$$

Eventually:

$$\sigma_{0S}(O\vec{z}_0) = ma^2\Big\{(\dot{\alpha}+\dot{\psi})\sin^2\theta + 2\left[\dot{\varphi}+(\dot{\alpha}+\dot{\psi})\cos\theta\right]\cos\theta + 4\dot{\alpha} + 2(\dot{\alpha}+\dot{\psi})\sin\theta\sin\psi$$
$$-\,2\dot{\theta}\cos\theta\cos\psi + 2\dot{\alpha}\sin\theta\sin\psi + (\dot{\alpha}+\dot{\psi})\sin^2\theta\Big\} \quad [8.31]$$

Comparing this expression with the Lagrange's equation [8.27] gives $\Gamma = \lambda$.

8.5.2. *Independent parameterization*

We transform the semi-holonomic constraint equation $\dot{\alpha} = \omega$ into the holonomic constraint equation $\alpha = \omega t$ by assuming that $\alpha = 0$ at the initial instant. We choose the following parameterization where the constraint equation $\alpha = \omega t$ is classified as primitive:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters: α, ψ, θ, φ.
- Primitive constraint equation: $\alpha = \omega t$.
- Retained parameters: ψ, θ, φ.
- No complementary constraint equation.

The kinetic energy of the system is obtained by making $\dot{\alpha} = \omega$ in [8.21]:

$$E^c_{0S} = ma^2\Big\{\dot{\theta}^2 + (\omega+\dot{\psi})^2\sin^2\theta + \left[\dot{\varphi}+(\omega+\dot{\psi})\cos\theta\right]^2 + 2\omega^2 +$$
$$2\omega(\omega+\dot{\psi})\sin\theta\sin\psi - 2\omega\dot{\theta}\cos\theta\cos\psi\Big\} \quad [8.32]$$

The potential due to the system's weight is always given by [8.22]: $\mathcal{V} = mga\cos\theta + const.$ By applying [8.2], we once again arrive at Lagrange's equations [8.28]–[8.30], which form three equations for the three unknowns ψ, θ and φ.

The advantage of this parameterization is that, as will be seen in section 9.6, it provides Painlevé's first integral.

8.6. Example: rigid body subjected to a double contact

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and a rigid body (S_1), composed of a disc (D) and a rod (T), in contact with a rotating plate and the axis $O\vec{z}_0$ (Figure 8.3). The disc (D) is of center C and radius $2a$. It is of negligible thickness, is homogeneous and has the mass m. The rod (T) is welded perpendicular to the disc at the point C, and has negligible mass and cross-sectional area. The whole system is subjected to the gravity field $-g\vec{z}_0$.

The disk (D) is in contact at a point I with a plate (S_2) of center O, and spinning at a uniform angular velocity ω around its vertical axis $O\vec{z}_0$. It is assumed that the disc rolls and pivots *without slipping* on (S_2). Through a suitable perfect mechanical joint, the rod (T) is forced to remain in contact with the axis $O\vec{z}_0$ at a point K that is *variable over* (T) *and over* $O\vec{z}_0$. The system studied is the rigid body (S_1) alone.

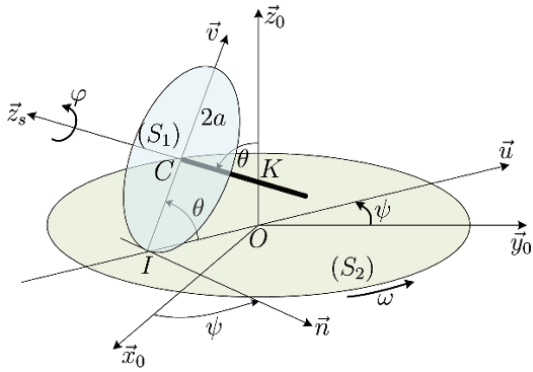


Figure 8.3. *Rigid body in contact with a rotating plate and the vertical axis*

8.6.1. *Preliminary analysis*

If the rigid body (S_1) is completely free in space, its position in R_0 is defined by six parameters that are taken to be equal to three cylindrical coordinates of the center C and to three Euler angles, ψ, θ and φ. The cylindrical coordinates of C are defined with respect to the coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and denoted by (r, α, z), with r being the polar radius of C, α the polar angle of C and z the elevation of C. The position of C in R_0 is thus written as

$$\overrightarrow{OC} = r\vec{e}_r(\alpha) + z\vec{z}_0, \qquad [8.33]$$

where $\vec{e}_r(\alpha)$ is the radial unit vector of the point C, a function of the angle α. Three Euler angles ψ, θ and φ are defined in the conventional manner:

– The precession angle ψ is defined as being the angle between $\vec{x}_0$ and the line of nodes $\vec{n}$, which is the intersection of the plane of the disc (D) with the plane $O\vec{x}_0\vec{y}_0$. This angle is measured around $\vec{z}_0$.

– The nutation angle θ is the angle between $\vec{z}_0$ and $\vec{z}_1$, measured around $\vec{n}$.

– The angle φ is the spin angle of (S_1) around $\vec{z}_1$.

The *a priori* position of (S_1) is, thus, defined by six parameters: r, α, z and ψ, θ, φ.

The rigid body (S_1) is subjected to two mechanical joints: the non-slip contact at I between the disc (D) and the plate (S_2), and the contact at K between the rod (T) and the axis $O\vec{z}_0$. We will now show that the set of joints can be expressed by four scalar relationships, which signifies that the position of (S_1) depends on two independent parameters alone. In order to obtain the constraint equations specific to each joint, we will study the joints *separately*.

1. The contact at K between the rod (T) and the axis $O\vec{z}_0$ is expressed by the fact that the three vectors $\overrightarrow{OC}, \vec{z}_1, \vec{z}_0$ are coplanar. In other words, their mixed product is zero: $(\overrightarrow{OC}, \vec{z}_1, \vec{z}_0) = 0$ (see Figure 8.4(a)), where we have represented the contact at K alone, excluding the contact at I.

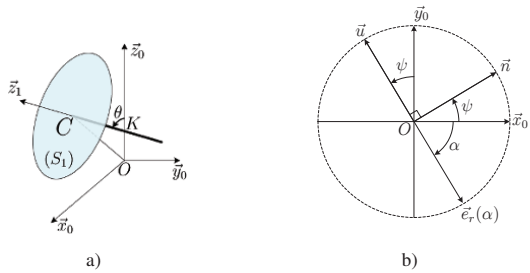


Figure 8.4. *Contact at point K*

Since

$$\left.\begin{array}{r}\overrightarrow{OC} = r\vec{e}_r(\alpha) + z\vec{z}_0 \\ \vec{z}_1 = -\sin\theta\vec{u} + \cos\theta\vec{z}_0\end{array}\right\} \quad \Rightarrow \quad (\overrightarrow{OC}, \vec{z}_1, \vec{z}_0) = -r\sin\theta\ \vec{e}_r(\alpha).\vec{n}$$

the condition for contact at K can be written as $\vec{e}_r(\alpha).\vec{n} = 0$, that is (see Figure 8.4(b))

$$\vec{e}_r(\alpha) = -\vec{u} \qquad \text{or} \qquad \alpha = \psi - \frac{\pi}{2} \qquad (\text{which implies } \dot{\vec{e}}_r = \dot{\psi}\vec{n}) \qquad [8.34]$$

The contact at K is, thus, expressed by a scalar relationship.

2. The contact at I consists of two conditions: the geometric contact condition (holonomic condition) and the kinematic condition of no-slip contact (semi-holonomic condition).

   The geometric contact at I between the disc (D) and the plate (S_2) is simply expressed by

$$z = a\sin\theta \qquad [8.35]$$

The no-slip condition at I is expressed by $\vec{V}_{21}(I) = \vec{0}$. Let us calculate the relative velocity $\vec{V}_{21}(I)$ by means of the composition formula for velocities [1.68]:

$$\vec{V}_{21}(I) = \vec{V}_{01}(I) - \vec{V}_{02}(I)$$

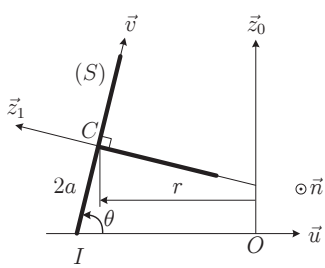


Figure 8.5. *Contact at point I*

The velocity $\vec{V}_{02}(I)$ can be easily obtained knowing that the distance OI has the value $r + 2a\cos\theta$ (Figure 8.5):

$$\vec{V}_{02}(I) = \omega(r + 2a\cos\theta)\vec{n}$$

Let us calculate $\vec{V}_{01}(I)$ using the formula $\vec{V}_{01}(I) = \vec{V}_{01}(C) + \vec{\Omega}_{01} \times \overrightarrow{CI}$. We have

$$\left.\begin{array}{l} \vec{\Omega}_{01} = \dot{\psi}\vec{z}_0 + \dot{\theta}\vec{n} + \dot{\varphi}\vec{z}_1 \\ \overrightarrow{CI} = -2a\vec{v} \end{array}\right\} \quad \Rightarrow \quad \vec{\Omega}_{01} \times \overrightarrow{CI} = 2a\left[(\dot{\varphi} + \dot{\psi}\cos\theta)\vec{n} - \dot{\theta}\vec{z}_1\right]$$

$$\overrightarrow{OC} \underset{[8.33]}{=} r\vec{e}_r(\alpha) + z\vec{z}_0 \underset{[8.35]}{=} r\vec{e}_r(\alpha) + 2a\sin\theta\vec{z}_0 \quad \Rightarrow \quad \vec{V}_{01}(C) = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r + 2a\dot{\theta}\cos\theta\vec{z}_0 \qquad [8.36]$$

Finally, taking into account $\vec{z}_1 = \cos\theta\vec{z}_0 - \sin\theta\vec{u}$, we arrive at

$$\vec{V}_{21}(I) = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r + \left\{2a\left[\dot{\varphi} + (\dot{\psi} - \omega)\cos\theta\right] - \omega r\right\}\vec{n} + 2a\dot{\theta}\sin\theta\vec{u} \qquad [8.37]$$

The condition $\vec{V}_{21}(I) = \vec{0}$ then gives

$$\dot{r}\vec{e}_r + r\dot{\vec{e}}_r + \left\{2a\left[\dot{\varphi} + (\dot{\psi} - \omega)\cos\theta\right] - \omega r\right\}\vec{n} + 2a\dot{\theta}\sin\theta\vec{u} = \vec{0} \qquad [8.38]$$

As the vectors $\vec{e}_r, \dot{\vec{e}}_r, \vec{n}, \vec{u}$ are in the plane $\vec{x}_0\vec{y}_0$, the no-slip condition at I is equivalent to two scalar relationships.

• If we take into account the two contacts at I and K at the same time (that is, [8.34] and [8.38]), then, by inserting [8.34] in [8.38], we obtain

$$\left[2a\dot{\varphi} + (r + 2a\cos\theta)(\dot{\psi} - \omega)\right]\vec{n} + (2a\dot{\theta}\sin\theta - \dot{r})\vec{u} = \vec{0}$$

$$\Leftrightarrow \quad \left\{\begin{array}{l} \dot{r} - 2a\dot{\theta}\sin\theta = 0 \\ 2a\dot{\varphi} + (r + 2a\cos\theta)(\dot{\psi} - \omega) = 0 \end{array}\right.$$

The first relationship amounts to saying that $r + 2a\cos\theta$ is equal to a constant, which we set to $2ka$ (k being a dimensionless constant):

$$OI = r + 2a\cos\theta = 2ka \qquad [8.39]$$

The constant $2ka$ is determined by the initial conditions: $2ka = r_0 + 2a\cos\theta_0$. The holonomic relationship [8.39] makes it possible to then recast the second relationship above as follows:

$$\dot{\varphi} + k(\dot{\psi} - \omega) = 0$$

This is a semi-holonomic relationship that we can integrate assuming the zero initial conditions on φ_0, ψ_0:

$$\varphi = k(\omega t - \psi) \qquad [8.40]$$

8.6.2. *Independent parameterization*

Given that the contact at K is expressed by the simple holonomic relationship [8.34], in this study we decide to count this contact from the beginning, in other words, to use relationship $\alpha = \psi - \dfrac{\pi}{2}$ from the beginning to eliminate angle α in favor of ψ. There will, thus, be five retained parameters for the problem: the two cylindrical coordinates (r, z) of center C and the three Euler angles ψ, θ, φ.

We choose the following parameterization:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters: $r, z, \psi, \theta, \varphi$.
- Primitive constraint equations:
  - [8.35]: $z = 2a\sin\theta$,
  - [8.39]: $r + 2a\cos\theta = 2ka$,
  - [8.40]: $\varphi = k(\omega t - \psi)$.
- Retained parameters: ψ, θ.
- No complementary constraint equation.

Let us calculate the kinetic energy of the system using König's formula: $E^c_{01} = E^c_{R^C_0 S_1} + \frac{1}{2}m\vec{V}^2_{01}(C)$, where $E^c_{R^C_0 S_1}$ denotes the kinetic energy calculated in the barycentric reference frame R^C_0 which has its origin in the center of mass C.

Knowing that the inertia matrix of the disc with radius $2a$ and mass m in any coordinate system with origin C and the third vector equal to $\vec{z}_1$ is $\mathbb{I}_{S_1}(C;\bullet,\bullet,\vec{z}_1) = \begin{bmatrix} A & & \\ & A & \\ & & C \end{bmatrix}$, with $C = 2A = 2ma^2$ (no confusion possible with the point C), we have

$$\begin{aligned} E^c_{R^C_0 S_1} &= \frac{1}{2}ma^2\left[(\dot\theta^2 + \dot\psi^2\sin^2\theta) + 2(\dot\varphi + \dot\psi\cos\theta)^2\right] \qquad \text{where } \dot\varphi = k(\omega - \dot\psi) \\ &= \frac{2}{1}ma^2\left\{(\dot\theta^2 + \dot\psi^2\sin^2\theta) + 2[k\omega - (k-\cos\theta)\dot\psi]^2\right\} \end{aligned} \qquad [8.41]$$

To calculate the velocity of the center C, let us develop [8.36] using relationship $\alpha = \psi - \frac{\pi}{2}$ and the primitive constraint equation [8.39]:

$$\begin{aligned} \vec{V}_{01}(C) &= \dot r\vec{e}_r + r\dot{\vec{e}}_r + 2a\dot\theta\cos\theta\vec{z}_0 \qquad \text{with} \begin{cases} r = 2a(k-\cos\theta) & \Rightarrow \dot r = 2a\dot\theta\sin\theta \\ \vec{e}_r(\alpha) = -\vec{u} & \Rightarrow \dot{\vec{e}}_r = \dot\psi\vec{n} \end{cases} \\ &= -2a\dot\theta\sin\theta\vec{u} + 2a(k-\cos\theta)\dot\psi\vec{n} + 2a\dot\theta\cos\theta\vec{z}_0 \end{aligned}$$

Thus

$$\frac{1}{2}m\vec{V}^2_{01}(C) = 2ma^2\left[\dot\theta^2 + (k-\cos\theta)^2\dot\psi^2\right] \qquad [8.42]$$

Hence

$$E^c_{01} = \frac{1}{2}ma^2\left\{5\dot\theta^2 + \left[6(k-\cos\theta)^2 + \sin^2\theta\right]\dot\psi^2 - 4k\omega(k-\cos\theta)\dot\psi + 2k^2\omega^2\right\} \qquad [8.43]$$

The potential due to the system's weight is

$$\mathcal{V} = mgz + const = 2mga\sin\theta + const \qquad [8.44]$$

The Lagrange's equations [8.2] give

$$\mathscr{L}_\theta: \quad 5\ddot\theta - \dot\psi^2(6k - 5\cos\theta)\sin\theta + 2k\omega\dot\psi\sin\theta + \frac{2g}{a}\cos\theta = 0 \qquad [8.45]$$

$$\mathscr{L}_\psi: \quad \frac{d}{dt}\left\{\left[6\left(k-\cos\theta\right)^2 + \sin^2\theta\right]\dot\psi - 2k\omega\left(k-\cos\theta\right)\right\} = 0 \qquad [8.46]$$

or

$$\left[6\left(k-\cos\theta\right)^2 + \sin^2\theta\right]\dot\psi - 2k\omega\left(k-\cos\theta\right) = const$$

8.6.3. *Reduced parameterization*

We keep the five primitive parameters as before, namely: $r, z, \psi, \theta, \varphi$, but we will change the parameterization by classifying certain constraint equations as complementary. The preliminary analysis carried out at the beginning shows that the constraint equations are obtained in the following order:

$$\left.\begin{array}{l}\text{[8.34]: contact at } K \\ \text{[8.35]: } z = 2a\sin\theta \Rightarrow \text{[8.38]: } \vec{V}_{21}(I) = \vec{0}\end{array}\right\} \begin{array}{l}\Rightarrow \text{[8.39]: } r + 2a\cos\theta = 2ka \\ \Rightarrow \text{[8.40]: } \varphi = k(\omega t - \psi)\end{array}$$

It follows from this that the complementary constraint equations cannot be freely chosen in an arbitrary manner:

- we can, for example, classify [8.40] alone – or [8.39] and [8.40] – as a complementary equation,
- on the contrary, we cannot count [8.39] as a complementary equation and leave [8.40] as a primitive equation.

In this section, we decide to take the following reduced parameterization:

REDUCED PARAMETERIZATION.

- Primitive parameters: $r, z, \psi, \theta, \varphi$.
- Primitive constraint equations:
  - [8.35]: $z = 2a\sin\theta$,
  - [8.39]: $r + 2a\cos\theta = 2ka$.
- Retained parameters: ψ, θ, φ.
- Complementary constraint equation: [8.40]: $\varphi = k(\omega t - \psi)$.

The kinetic energy of the rigid body S_1 with respect to R_0 is always calculated using König's formula: $E^c_{01} = E^c_{R_0^C S_1} + \dfrac{1}{2}m\vec{V}^2_{01}(C)$, where $\frac{1}{2}m\vec{V}^2_{01}(C)$ was obtained in [8.42] and $E^c_{R_0^C S_1}$ in $[8.41]_1$:

$$E^c_{R_0^C S_1} = \frac{1}{2}ma^2\left[(\dot\theta^2 + \dot\psi^2\sin^2\theta) + 2(\dot\varphi + \dot\psi\cos\theta)^2\right]$$

Hence

$$E^c_{01} = \frac{1}{2}ma^2\left\{5\dot\theta^2 + \left[4(k-\cos\theta)^2 + \sin^2\theta\right]\dot\psi^2 + 2(\dot\varphi + \dot\psi\cos\theta)^2\right\}$$

The potential of the rigid body S_1 is always given by [8.44]. The Lagrange's equations [8.6] give three equations, with λ denoting the multiplier corresponding to the complementary constraint equation:

$$\begin{array}{ll}\mathscr{L}_\psi: & ma^2\dfrac{d}{dt}\left\{\left[4(k-\cos\theta)^2 + \sin^2\theta\right]\dot\psi + 2(\dot\varphi + \dot\psi\cos\theta)\cos\theta\right\} = k\lambda \\ \mathscr{L}_\theta: & 5\ddot\theta - \sin\theta\left[(4k - 5\cos\theta)\dot\psi^2 - 2\dot\psi\dot\varphi\right] + 2\dfrac{g}{a}\cos\theta = 0 \\ \mathscr{L}_\varphi: & 2ma^2\dfrac{d}{dt}(\dot\varphi + \dot\psi\cos\theta) = \lambda\end{array}$$

Taking into account the complementary constraint equation $\varphi = k(\omega t - \psi)$, we arrive at three equations for three unknowns ψ, θ and λ.

• Let us find the mechanical significance of the multiplier λ. By denoting $\vec{F}_I$ the force at I exerted by the plate S_2 on the rigid body S_1, we can write the VP of the constraint force exerted by S_2 on S_1 as

$$\mathcal{P}^*(\vec{F}_I) = \vec{F}_I.\vec{V}^*_{21}(I)$$

The relative VV $\vec{V}^*_{21}(I)$ is calculated by the composition formula for velocities [4.49]:

$$\vec{V}^*_{21}(I) = \vec{V}^*_{01}(I) - \vec{V}^*_{02}(I)$$

As the rotation tensor $\bar{\bar{Q}}_{02}$ does not depend on the retained parameters, relationship [4.30] gives $\vec{\Omega}^*_{02} = \vec{0}$. Hence, $\vec{V}^*_{02}(I) = \vec{0}$ and thus $\vec{V}^*_{21}(I) = \vec{V}^*_{01}(I)$, which is predictable from [4.54].

Let us calculate $\vec{V}^*_{01}(I)$ using formula [4.35]: $\vec{V}^*_{01}(I) = \vec{V}^*_{01}(C) + \vec{\Omega}^*_{01} \times \overrightarrow{CI}$. We have

$$\begin{aligned} \overrightarrow{OC} &\underset{[8.33]}{=} r\vec{e}_r(\alpha) + z\vec{z}_0 \qquad \text{with} \begin{cases} \alpha = \psi - \frac{\pi}{2} \quad \text{or} \quad \vec{e}_r(\alpha) = -\vec{u} \\ [8.35]: z = 2a\sin\theta \\ [8.39]: r + 2a\cos\theta = 2ka \end{cases} \\ &= 2a(\cos\theta - k)\vec{u} + 2a\sin\theta\vec{z}_0 \end{aligned}$$

Hence

$$\begin{aligned} \vec{V}^*_{01}(C) &= \frac{\overrightarrow{\partial C}}{\partial\psi}\dot{\psi}^* + \frac{\overrightarrow{\partial C}}{\partial\theta}\dot{\theta}^* \\ &= 2a(k - \cos\theta)\vec{n}\dot{\psi}^* + 2a\underbrace{(-\sin\theta\vec{u} + \cos\theta\vec{z}_0)}_{\vec{z}_1}\dot{\theta}^* \end{aligned}$$

On the other hand

$$\left.\begin{aligned} &\vec{\Omega}^*_{01} = \dot{\psi}^*\vec{z}_0 + \dot{\theta}^*\vec{n} + \dot{\varphi}^*\vec{z}_1 \\ &\overrightarrow{CI} = -2a\vec{v} \end{aligned}\right\} \Rightarrow \quad \vec{\Omega}^*_{01} \times \overrightarrow{CI} = 2a\left[(\dot{\varphi}^* + \dot{\psi}^*\cos\theta)\vec{n} - \dot{\theta}^*\vec{z}_1\right]$$

Finally, we arrive at

$$\vec{V}^*_{21}(I) = 2a(\dot{\varphi}^* + k\dot{\psi}^*)\vec{n}$$

Hence

$$\mathscr{P}^*(\vec{F}_I) = 2a(\dot{\varphi}^* + k\dot{\psi}^*)\vec{F}_I.\vec{n}$$

By identifying this expression with $\mathscr{P}^*(\vec{F}_I) = \lambda(\dot{\varphi}^* + k\dot{\psi}^*)$, we find

$$\lambda = 2a\ \vec{F}_I.\vec{n}$$

Thus, the multiplier λ appears as the moment of the force $\vec{F}_I$ about the axis $C\vec{z}_1$.

• We can also find the mechanical meaning of the multiplier using Newtonian mechanics. By resolving the force $\vec{F}_I$ into components relative to the basis $(\vec{n}, \vec{u}, \vec{z}_0)$, we can see that

$$\mathcal{M}_{ext\to 1}(C\vec{z}_1) = 2a\ \vec{F}_I.\vec{n}$$

On the other hand,

$$\mathcal{M}_{ext\to 1}(C\vec{z}_1) = \delta_{01}(C\vec{z}_1) = \underbrace{2ma^2}_{C}\frac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta) = \lambda$$

We do indeed once again find $\lambda = 2a\ \vec{F}_I.\vec{n}$.

9

First Integrals

Consider a mechanical system $\mathcal{S}$ moving in a Galilean reference frame R_g. The parameterization chosen for $\mathcal{S}$ may or may not be independent, that is, it may or may not contain complementary constraint equations. The retained parameters for $\mathcal{S}$ are $q \equiv (q_1, \ldots, q_n)$ and t.

Definition. *A first integral* of the motion of $\mathcal{S}$ (or the first integral of $\mathcal{S}$) is, by definition, a first-order differential equation in q and satisfied by the solutions $t \mapsto q(t)$ of the equations of motion of $\mathcal{S}$. A first integral is of the form

$$\boxed{\psi(q(t), \dot{q}(t), t) = const} \qquad [9.1]$$

where ψ is a scalar function of $(2n+1)$ variables $(q, \dot{q}, t)$ and $const$ denotes a constant that usually depends on the initial conditions $q(t_0)$ and $\dot{q}(t_0)$ at t_0.

By definition, the constraint equations, if they exist, are the first integrals of the motion.

• The first integrals that we will see in the sequel are derived from the Lagrange's equations obtained in Chapter 8 in the case of perfect joints. Consequently, we continue to adopt (as was done in Chapter 8) convention [6.1] according to which, the Galilean reference frame R_g being known, we choose the common reference frame R_0 equal to R_g:

$$\boxed{R_0 = R_g}$$

This choice allows the pair (R_g, R_0) to *automatically satisfy hypothesis* [2.33], which is required for the definition of perfect joints.

• NOTATION CONVENTION. To simplify the writing in this chapter, the kinetic energy and the potential will be written without subscript: E^c, $\mathcal{V}$, instead of $E^c_{R_g\mathcal{S}}$, $\mathcal{V}_{R_g}$.

9.1. Painlevé's first integral

9.1.1. *Painlevé's lemma*

Painlevé's first integral involves the parameterized kinetic energy $E^c(q, \dot{q}, t)$ of the system $\mathcal{S}$ with respect to a reference frame R_g defined by [2.54] and its decomposition [2.57]:

$$E^c = E^{c(2)} + E^{c(1)} + E^{c(0)}$$

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

where $E^{c(2)}$, $E^{c(1)}$ and $E^{c(0)}$ are defined as the parts of E^c which are respectively of second, first and zero degree with respect to the derivatives $\dot{q}_i$. Let us first establish the following preliminary result:

Painlevé's lemma. We have the following identity:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(E^{c(2)} - E^{c(0)}\right) = \sum_{i=1}^{n} Q_i \dot{q}_i - \frac{\partial E^c}{\partial t}} \qquad [9.2]$$

where Q_i is the ith generalized force applied on $\mathcal{S}$. This identity is general, in the sense that it is valid regardless of the joints (perfect or not) and regardless of the efforts applied to the system (whether derivable from a potential or not).

PROOF. Let us begin with the Lagrange's equations [6.2]:

$$\forall t, \forall i \in [1, n], \qquad \frac{d}{dt}\frac{\partial E^c}{\partial \dot{q}_i} - \frac{\partial E^c}{\partial q_i} = Q_i$$

By multiplying the Lagrange's equation number i by $\dot{q}_i$ and by summing the equalities thus obtained over i, we arrive at what is called *Painlevé's combination*:

$$\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E^c}{\partial \dot{q}_i}\dot{q}_i - \sum_{i=1}^{n} \frac{\partial E^c}{\partial q_i}\dot{q}_i = \sum_{i=1}^{n} Q_i \dot{q}_i \qquad [9.3]$$

which, in fact, is simply the PVP written with $\dot{q}$ instead of $\dot{q}^*$ (see [8.9]).

Let us transform the left-hand side of [9.3] using the decomposition $\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E^c}{\partial \dot{q}_i}\dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial E^c}{\partial \dot{q}}\dot{q}_i\right) - \frac{\partial E^c}{\partial \dot{q}_i}\ddot{q}_i$:

$$\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E^c}{\partial \dot{q}_i}\dot{q}_i - \sum_{i=1}^{n} \frac{\partial E^c}{\partial q_i}\dot{q}_i = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{n} \frac{\partial E^c}{\partial \dot{q}}\dot{q}_i - \left(\sum_{i=1}^{n} \frac{\partial E^c}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial E^c}{\partial \dot{q}_i}\ddot{q}_i\right) \qquad [9.4]$$

homogeneous functions of the generalized velocities of order n = 2,1,0.

Further, since the velocity parameterization is affine in $\dot{q}$, the kinetic energy E^c is of second order in $\dot{q}$: $E^c = E^{c(2)} + E^{c(1)} + E^{c(0)}$, where $E^{c(k)}$ is a (positively) homogeneous function of $\dot{q}_j$ of order k. Recalling that a function $\mathbb{R}^n \ni x \mapsto f(x)$ (positively) homogeneous of order k satisfies the Euler identity $\sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(x) = kf(x)$, we arrive at

$$\sum_{i=1}^{n} \frac{\partial E^c}{\partial \dot{q}_i}\dot{q}_i = 2 \times E^{c(2)} + 1 \times E^{c(1)} + 0 \times E^{c(0)} = 2E^{c(2)} + E^{c(1)}$$

On the other hand, from the identity $\frac{\mathrm{d}E^c}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial E^c}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial E^c}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial E^c}{\partial t}$, we get

$$\sum_{i=1}^{n} \frac{\partial E^c}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial E^c}{\partial \dot{q}_i}\ddot{q}_i = \frac{\mathrm{d}E^c}{\mathrm{d}t} - \frac{\partial E^c}{\partial t} \qquad [9.5]$$

Thus, [9.4] becomes

$$\begin{aligned}\sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E^c}{\partial \dot{q}_i}\dot{q}_i - \sum_{i=1}^{n} \frac{\partial E^c}{\partial q_i}\dot{q}_i &= \frac{\mathrm{d}}{\mathrm{d}t}\left(2E^{c(2)} + E^{c(1)}\right) - \frac{\mathrm{d}E^c}{\mathrm{d}t} + \frac{\partial E^c}{\partial t} \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(E^{c(2)} - E^{c(0)}\right) + \frac{\partial E^c}{\partial t}\end{aligned}$$

By inserting this result in [9.3], we obtain the desired result. ■

REMARK. In the above proof, depending on the context, E^c is either the parameterized kinetic energy $E^c(q, \dot{q}, t)$, defined by [2.54] and function of the $2n + 1$ parameters $(q, \dot{q}, t)$, or it is the conventional kinetic energy, composite function of time via a given motion $t \mapsto q(t)$. ■

9.1.2. *Painlevé's first integral*

Theorem. Painlevé's first integral

HYPOTHESES:

(i) The reference frame R_g is Galilean and we choose $R_0 = R_g$ as per convention [6.1].

(ii) The joints are *perfect*.

(iii) The given efforts on $\mathcal{S}$ *are derivable, in* R_g, *from a potential* $\mathcal{V}(q, t)$ (using the chosen parameterization).

(iv) The parameterization is such that *the Lagrangian* $\mathbb{L}(q, \dot{q}, t) = E^c(q, \dot{q}, t) - \mathcal{V}(q, t)$ *is time independent*:

$$\frac{\partial \mathbb{L}}{\partial t} = \frac{\partial}{\partial t}(E^c - \mathcal{V}) = 0$$

(a sufficient, but not necessary, condition for this hypothesis is that $\dfrac{\partial E^c}{\partial t} = \dfrac{\partial \mathcal{V}}{\partial t} = 0$).

(v) The *complementary* constraint equation written in differential form is *homogeneous*:

$$\sum_{i=1}^{n} \alpha_{hi}\, \dot{q}_i + \underbrace{\beta_h}_{=0} = 0, \qquad h \in [1, \ell]$$

The system then has *Painlevé's first integral*:

$$\boxed{E^{c(2)} - E^{c(0)} + \mathcal{V} = const} \qquad [9.6]$$

FIRST PROOF. Owing to hypothesis (ii), the ith generalized force Q_i is given by (see the proof of Lagrange's equation [8.6]):

$$Q_i = D_i + L_i = D_i + \sum_{h=1}^{\ell} \lambda_h\, \alpha_{hi}$$

where D_i (respectively, L_i) is the ith generalized force corresponding to the given efforts (respectively, the constraint efforts). Consequently, the sum $\sum_{i=1}^{n} Q_i \dot{q}_i$ in identity [9.2] can be written as

$$\sum_{i=1}^{n} Q_i \dot{q}_i = \sum_{i=1}^{n} D_i \dot{q}_i + \sum_{i=1}^{n} \left(\sum_{h=1}^{\ell} \lambda_h\, \alpha_{hi} \right) \dot{q}_i \qquad [9.7]$$

On the one hand, hypothesis (iii) implies that $\exists \mathcal{V}(q,t), D_i = -\dfrac{\partial \mathcal{V}}{\partial q_i}, \forall i$. Hence

$$\begin{aligned}\sum_{i=1}^{n} D_i \dot{q}_i &= -\sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial q_i}\dot{q}_i \\ &= -\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} + \frac{\partial \mathcal{V}}{\partial t} \text{ noting that } \frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t} = \sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial q_i}\dot{q}_i + \frac{\partial \mathcal{V}}{\partial t}\end{aligned} \qquad [9.8]$$

On the other hand, the last term in [9.7] can be recast as follows:

$$\sum_{i=1}^{n}\left(\sum_{h=1}^{\ell} \lambda_h \alpha_{hi}\right)\dot{q}_i = \sum_{h=1}^{\ell} \lambda_h \underbrace{\left(\sum_{i=1}^{n} \alpha_{hi}\dot{q}_i\right)}_{=-\beta_h} = -\sum_{h=1}^{\ell} \lambda_h \beta_h \underset{\substack{\text{hypothesis}\\(\text{v})}}{=} 0 \qquad [9.9]$$

Taking into account [9.7] and [9.8], identity [9.2] becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E^{c(2)} - E^{c(0)} + \mathcal{V}\right) = -\frac{\partial}{\partial t}\left(E^c - \mathcal{V}\right)$$

The right-hand side vanishes because of hypothesis (iv) above. ■

SECOND PROOF. Below is another proof for Painlevé's first integral, which may be instructive. This proof makes use of the hypothesis of perfect joints without using Lagrange multipliers.

As in the first proof, we begin by writing $Q_i = D_i + L_i$, which gives

$$\sum_{i=1}^{n} Q_i \dot{q}_i = \sum_{i=1}^{n} D_i \dot{q}_i + \sum_{i=1}^{n} L_i \dot{q}_i$$

The sum $\sum_{i=1}^{n} D_i \dot{q}_i$ is transformed as in the first proof and we will examine, here, the sum $\sum_{i=1}^{n} L_i \dot{q}_i$. Consider the particular VVF defined with $\dot{q}_i^* = \dot{q}_i$:

$$\vec{V}^*(p) = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i^* = \sum_{i=1}^{n} \frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i \qquad [9.10]$$

According to hypothesis (v), the $\dot{q}_i$ satisfy $\sum_{i=1}^{n} \alpha_{hi}\,\dot{q}_i = 0,\ \forall h \in [1,\ell]$. Consequently, according to hefinition [7.2], the VVF [9.10] is compatible with the joints of the system.

Since the joints are perfect (hypothesis (ii)), the VP of the constraint efforts in the compatible VVF [9.10] is zero:

$$0 = \mathscr{P}^*_{R_g}(\mathcal{F}_{\text{constraint}\to\mathcal{S}}) = \sum_{i=1}^{n} L_i \dot{q}_i^* = \sum_{i=1}^{n} L_i \dot{q}_i$$

This is the same result as in [9.9]. We conclude as in the first proof. ■

Instead of memorizing the hypotheses for Painlevé's first integral, it would be better to start from identity [9.2] and see, in each given problem, which hypotheses will enable one to optimally simplify [9.2].

For the same mechanical system $\mathcal{S}$, there may or may not exist a Painlevé's first integral depending upon the chosen parameterization. Indeed, the expressions for the kinetic energy E^c and the potential $\mathcal{V}$ depend on the parameterization used and it may be that with a certain unnatural parameterization, they will not satisfy hypothesis (iv), for instance.

9.2. The energy integral: conservative systems

The following theorem can be obtained as a particular case of Painlevé's first integral.

Theorem and definition. The energy integral.
HYPOTHESES:

(i) The reference frame R_g is *Galilean* and we choose $R_0 = R_g$ according to convention [6.1].

(ii) $\dfrac{\partial_{R_g}\overrightarrow{O_gP}}{\partial t} = \vec{0}$ (where O_g is a point fixed in R_g).

(iii) The joints are *perfect*.

(iv) The given efforts on $\mathcal{S}$ are derivable, in R_g, from a *time-independent* potential $\mathcal{V}(q)$ (this hypothesis is a little stronger than Painlevé's first integral).

(v) The *complementary* constraint equations written in differential form are *homogeneous*:

$$\sum_{i=1}^{n} \alpha_{hi}\,\dot{q}_i + \underbrace{\beta_h}_{=0} = 0, \qquad h \in [1, \ell]$$

Under these hypotheses, we have *the energy integral*:

$$\boxed{E^c + \mathcal{V} = const} \qquad [9.11]$$

The sum $E^c + \mathcal{V}$ is called *the mechanical energy* of system $\mathcal{S}$. We say that the system is *conservative*.

FIRST PROOF. Hypothesis (ii) makes it possible to apply [2.59] and to have

$$E^{c(0)} = E^{c(1)} = 0 \qquad \text{and} \qquad E^c = E^{c(2)}(q, \dot{q}) \text{ independent of } t$$

Since $\dfrac{\partial E^c}{\partial t} = 0$ and since, according to (iv), $\dfrac{\partial \mathcal{V}}{\partial t} = 0$, the hypothesis $\dfrac{\partial}{\partial t}(E^c - \mathcal{V}) = 0$ for Painlevé's first integral is satisfied.

Thus, the set of hypotheses adopted, (i)–(v), yields Painlevé's first integral [9.6]:

$$\underbrace{E^{c(2)}}_{=E^c} - \underbrace{E^{c(0)}}_{=0} + \mathcal{V} = const \qquad \blacksquare$$

SECOND PROOF. Here is a direct proof for [9.11] without using Painlevé's first integral and [2.59]. Differentiating the kinetic energy $E^c = \dfrac{1}{2}\displaystyle\int_{\mathcal{S}} \vec{V}^2_{R_g}(p,t)\mathrm{d}m$ with respect to time gives

$$\begin{aligned}
\frac{\mathrm{d}E^c}{\mathrm{d}t} &= \int_{\mathcal{S}} \vec{V}_{R_g}(p,t).\vec{\Gamma}_{R_g}(p,t)\,dm \qquad \text{where, from hypothesis (ii), } \vec{V}_{R_g}(p,t) = \sum_i \frac{\overrightarrow{\partial P}}{\partial q_i}\dot{q}_i \\
&= \sum_i \int_{\mathcal{S}} \frac{\overrightarrow{\partial P}}{\partial q_i}.\vec{\Gamma}_{R_g}(p,t)\,\mathrm{d}m\,\dot{q}_i = \sum_i C_i\dot{q}_i \qquad \text{according to definition [5.40] of } C_i \\
&= \sum_i Q_i\dot{q}_i \qquad \text{as } C_i = Q_i \text{ according to Lagrange's equation [6.2]}
\end{aligned}$$

[9.12]

The decomposition [5.19], $Q_i = D_i + L_i$, gives

$$\sum_i Q_i \dot{q}_i = \sum_i D_i \dot{q}_i + \sum_i L_i \dot{q}_i$$

where, according to hypothesis (iv):

$$\sum_i D_i \dot{q}_i = -\sum_i \frac{\partial \mathcal{V}}{\partial q_i} \dot{q}_i = -\frac{\mathrm{d}\mathcal{V}}{\mathrm{d}t}$$

Inserting these last results into [9.12] gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E^c + \mathcal{V}\right) = \sum_i L_i \dot{q}_i \qquad [9.13]$$

Using the same reasoning in the proof for Painlevé's first integral, it can be shown that the sum $\sum_i L_i \dot{q}_i$ is zero. ■

REMARK. The energy integral in Newtonian mechanics differs slightly from the first integral [9.11]. In Newtonian mechanics:

- There is no need for hypothesis $\dfrac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} = \vec{0}$.
- The hypotheses do not involve the parameterization, unlike hypotheses (i), (ii), (iv) and (v) adopted here for analytical mechanics.
- The first integral brings into play the potential energy $E^p(q(t), t)$ instead of the time-independent potential $\mathcal{V}(q)$.
- The proof of the first integral is carried out starting from the kinetic energy theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[E^c_{R_g\mathcal{S}}(t) + E^p_{R_g}(t)\right] = \mathscr{P}_{R_g}(\bar{\mathcal{F}}_{\to\mathcal{S}}, t),$$

  where $\bar{\mathcal{F}}_{\to\mathcal{S}}$ denotes the efforts that are not derivable from a potential energy; in this case these are the constraint efforts. The previous relationship resembles [9.13]. ■

9.2.1. *Energy considerations in addition to the energy integral*

Definition. A system of efforts $\mathcal{F}_{\to\mathcal{S}}$ applied to the system $\mathcal{S}$ is *dissipative* if, in *any* motion of $\mathcal{S}$, the *real* power of $\mathcal{F}_{\to\mathcal{S}}$ with respect to a Galilean reference frame is negative or zero:

$$\mathscr{P}\left(\mathcal{F}_{\to\mathcal{S}}\right) \leq 0$$

Theorem on the inequality of mechanical energy.

HYPOTHESES:

(i) The reference frame R_g is *Galilean* and we choose $R_0 = R_g$ according to convention [6.1].

(ii) $\dfrac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} = \vec{0}$.

(iii) There are two types of efforts acting upon $\mathcal{S}$:

- some are derivable, in R_g, from a *time-independent* potential $\mathcal{V}(q)$;
- others, denoted $\mathcal{F}_{\text{dissipative}\to\mathcal{S}}$, are *dissipative*.

(iv) The *complementary* constraint equations written in differential form are *homogeneous*:

$$\sum_{i=1}^{n} \alpha_{hi}\,\dot{q}_i + \underbrace{\beta_h}_{=0} = 0, \qquad h \in [1, \ell]$$

Thus, in *any* motion $t \mapsto q(t)$ and for any $t \geq t_0$, the mechanical energy of the system decreases:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\,(E^c + \mathcal{V}) \leq 0} \quad \text{or} \quad \boxed{E^c(q(t), \dot{q}(t)) + \mathcal{V}(q(t)) \leq E^c(q(t_0), \dot{q}(t_0)) + \mathcal{V}(q(t_0))}$$

PROOF. The reasoning is the same as in the second proof for the energy integral [9.11], except that here relationship [9.13] *ibid* becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(E^c + \mathcal{V}) = \mathscr{P}(\mathcal{F}_{\text{dissipative}\to\mathcal{S}}) \leq 0 \qquad \blacksquare$$

9.3. Example: disk rolling on a suspended rod

Consider a Galilean reference frame $R_g = R_0$ endowed with the orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and a system $\mathcal{S}$ made up of two rigid bodies (Figure 9.1):

- A disk S_1 with radius R, mass m and center C.
- A rod $S_2 = AB$, with length $2L$, mass M and center of mass J.

The system is moving in the plane $O\vec{x}_0\vec{y}_0$ under the gravity field $-g\vec{y}_0$. The disk S_1 rolls along the rod S_2 without sliding, the point of contact being denoted by I. The rod S_2 is suspended from two parallel, identical wires AD and BE, with length l, with no mass and with the attachment points D, E fixed in R_0, such that the motion of the rod is a circular translation.

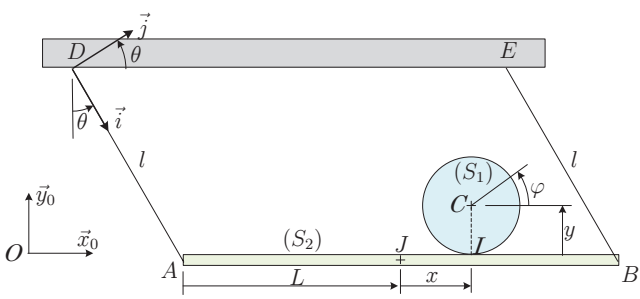


Figure 9.1. *Disk rolling along a suspended rod*

The position of the rod S_2 in R_0 is defined by the angle θ formed by the wires AD, BE and $\vec{y}_0$ (Figure 9.1). The position of the disk S_1 is defined relative to S_2, by the coordinates x, y of center C with respect to the coordinate system $(J; \vec{x}_0, \vec{y}_0)$ and the angle φ between $\vec{x}_0$ and a given radius of the disk.

The contact between the disk and the rod is expressed by $y = R$. The no-slip condition at point I is expressed by $\vec{V}_{21}(I) = \vec{0}$, where the index 2 denotes the reference frame R_2 defined by S_2. The slip velocity $\vec{V}_{21}(I)$ is calculated using the formula

$$\vec{V}_{21}(I) = \vec{V}_{21}(C) + \vec{\Omega}_{21} \times \overrightarrow{CI}$$

where $\overrightarrow{CI} = -R\vec{y}_0$ and $\vec{\Omega}_{21} = \dot{\varphi}\vec{z}_0$. As concerns $\vec{V}_{21}(C)$, it can be calculated using definition [1.47]:

$$\vec{V}_{21}(C) = \frac{d_{R_2}\overrightarrow{JC}}{dt} \equiv \bar{\bar{Q}}_{02}.\frac{d}{dt}\left(\bar{\bar{Q}}_{20}.\overrightarrow{JC}\right)$$

knowing that J is a fixed point in R_2. Since the rod S_2 is in translation with respect to R_0, the rotation tensor $\bar{\bar{Q}}_{02}$ is the identity tensor and, consequently

$$\vec{V}_{21}(C) = \frac{d\overrightarrow{JC}}{dt} = \dot{x}\vec{x}_0 \qquad \text{knowing that } \overrightarrow{JC} = x\vec{x}_0 + R\vec{y}_0$$

Thus, the slip velocity is $\vec{V}_{21}(I) = (\dot{x} + R\dot{\varphi})\vec{x}_0$ and the condition for no-slip contact at point I is finally expressed by $\dot{x} + R\dot{\varphi} = 0$.

Given the above results, we choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: θ, x, y, φ.
- Primitive constraint equation: $y = R$.
- Retained parameters: θ, x, φ.
- Complementary constraint equation: $\dot{x} + R\dot{\varphi} = 0$.

The kinetic energy of the system is

$$E^c = E^c_{01} + E^c_{02} \text{ with } \begin{cases} E^c_{01} = E^c_{R_0^C S_1} + \dfrac{1}{2}m\vec{V}^2_{01}(C) \\ E^c_{02} = \dfrac{1}{2}M\vec{V}^2_{02}(J) \end{cases}$$

where $E^c_{R_0^C S_1}$ denotes the kinetic energy with respect to the barycentric reference frame R_0^C, which has its origin in the center of mass C of S_1. By defining the vector $\vec{i}$ as being the unit vector parallel to $\overrightarrow{DA}$ and the vector $\vec{j} \equiv \vec{z}_0 \times \vec{i}$ (Figure 9.1), we have

$$\begin{cases} E^c_{R_0^C S_1} = \dfrac{1}{2}\dfrac{1}{2}mR^2\dot{\varphi}^2 \\ \overrightarrow{DC} = \overrightarrow{DA} + \overrightarrow{AI} + \overrightarrow{IC} = \ell\vec{i} + (L+x)\vec{x}_0 + R\vec{y}_0 \Rightarrow \vec{V}_{01}(C) = \ell\dot{\theta}\vec{j} + \dot{x}\vec{x}_0 \\ \overrightarrow{DJ} = \overrightarrow{DA} + \overrightarrow{AJ} = \ell\vec{i} + L\vec{x}_0 \qquad\qquad \Rightarrow \vec{V}_{02}(J) = \ell\dot{\theta}\vec{j} \end{cases}$$

We thus obtain

$$E^c = \frac{1}{2}(M+m)\ell^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 + m\ell\dot{\theta}\dot{x}\cos\theta + \frac{1}{4}mR^2\dot{\varphi}^2$$

The potential comes from the weight of the two rigid bodies:

$$\mathcal{V} = -Mg\ell\cos\theta - mg\ell\cos\theta + mgR + const = -(M+m)g\ell\cos\theta + const$$

• The Lagrange's equation [8.6] can be written as follows, with λ denoting the multiplier associated with the complementary constraint equation $\dot{x} + R\dot{\varphi} = 0$:

$$\begin{aligned}
\mathscr{L}_\theta :\ & (M+m)\ell\ddot{\theta} + m\ddot{x}\cos\theta + (M+m)g\sin\theta &= 0\\
\mathscr{L}_x :\ & m\ddot{x} + m\ell\frac{d}{dt}(\dot{\theta}\cos\theta) &= \lambda\\
\mathscr{L}_\varphi :\ & \frac{1}{2}mR\ddot{\varphi} &= \lambda R
\end{aligned}$$

Now taking into account the complementary constraint equation $\dot{x}+R\dot{\varphi} = 0$, we can eliminate $\dot{x}$ in favor of $\dot{\varphi}$ and arrive at a system of three equations for the three unknowns θ, φ, λ:

$$\begin{aligned}
& (M+m)\ell\ddot{\theta} - mR\ddot{\varphi}\cos\theta + (M+m)g\sin\theta &= 0\\
& -mR\ddot{\varphi} + m\ell\frac{d}{dt}(\dot{\theta}\cos\theta) &= \lambda\\
& \frac{1}{2}m\ddot{\varphi} &= \lambda
\end{aligned} \qquad [9.14]$$

By eliminating λ from equations $[9.14]_2$ and $[9.14]_3$, we arrive at

$$-\ddot{x} = R\ddot{\varphi} = \frac{2\ell}{3}\frac{d}{dt}(\dot{\theta}\cos\theta) \qquad [9.15]$$

Using this relationship, we can transform $[9.14]_1$ into a differential equation in θ:

$$\left(1 - \frac{2m}{3(M+m)}\cos^2\theta\right)\ddot{\theta} + \frac{2m}{3(M+m)}\dot{\theta}^2\sin\theta\cos\theta + \frac{g}{\ell}\sin\theta = 0 \qquad [9.16]$$

Solving this equation provides θ as a function of time. Relationship [9.15] then enables one to derive the functions $x(t)$ and $\varphi(t)$.

• The hypotheses for the energy integral [9.11] are satisfied here:

– the position of a current particle of the system is not explicitly time dependent: $\dfrac{\partial_{R_g}\overrightarrow{OP}}{\partial t} = \vec{0}$,

– the joints are perfect,

– the given efforts on the system are derivable in R_g from a time-independent potential $\mathcal{V}$,

– the complementary constraint equation $\dot{x} + R\dot{\varphi} = 0$ is homogeneous.

Thus, we have the energy integral $E^c + \mathcal{V} = const$:

$$\frac{1}{2}(M+m)\ell^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 + m\ell\dot{\theta}\dot{x}\cos\theta + \frac{1}{4}mR^2\dot{\varphi}^2 - (M+m)g\ell\cos\theta = const$$

or, with $\dot{x} = -R\dot{\varphi}$:

$$\frac{1}{2}(M+m)\ell^2\dot{\theta}^2 + \frac{3}{4}mR^2\dot{\varphi}^2 - m\ell R\dot{\theta}\dot{\varphi}\cos\theta - (M+m)g\ell\cos\theta = const \qquad [9.17]$$

This first integral may also be obtained through the following combination of Lagrange's equations:

$$[9.17] = \int\left([9.14]_1 \times \ell\dot{\theta} + [9.15] \times \frac{3mR}{2}\dot{\varphi}\right)dt$$

In other problems that concern first integrals, it is not always easy to arrive at these first integrals through a combination of Lagrange's equations.

9.4. Example: particle on a rotating hoop

Consider the example of the particle on a rotating hoop, which is studied in section 8.3 using the independent parameterization. The hypotheses for Painlevé's first integral [9.6] are satisfied. Indeed:

– The joint between the particle p and the hoop (C) is frictionless and, thus, perfect. The joint between the hoop (C) and the engine that makes it rotate at a constant velocity is also perfect. Indeed, the VP of the engine torque exerted on the hoop is $\mathscr{P}^*_{\text{engine}\to C} = \Gamma\vec{z}_0.\vec{\Omega}^*_{R_0C} = 0$ because $\vec{\Omega}^*_{R_0C} = \vec{0}$.

– The kinetic energy of the system consisting of the particle and the hoop is

$$E^c = \frac{1}{2}ma^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + \frac{1}{2}I\omega^2$$

The potential of the system is

$$\mathcal{V} = mga\cos\theta + const$$

Thus, the hypothesis $\dfrac{\partial}{\partial t}(E^c - \mathcal{V}) = 0$ is satisfied.

– Finally, there is no complementary constraint equation.

Painlevé's first integral [9.6] is written as

$$\dot{\theta}^2 - \omega^2\sin^2\theta + 2\frac{g}{a}\cos\theta = const$$

a relationship that is identical to [8.12], which was directly obtained through the transformation of the Lagrange's equation [8.11].

9.5. Example: a rigid body connected to a rotating rod by a spherical joint (no. 1)

Consider the rigid body connected by a spherical joint to a rotating rod, which is studied in section 8.4 (Figure 8.2). The equations for α and φ, [8.23] and [8.26], immediately give two first integrals:

$$2(\dot{\alpha}+\dot{\psi})sin^2\theta+2\left[\dot{\varphi}+(\dot{\alpha}+\dot{\psi})\cos\theta\right]\cos\theta+4\dot{\alpha}+(4\dot{\alpha}+2\dot{\psi})\sin\theta\sin\psi-2\dot{\theta}\cos\theta\cos\psi = const \quad [9.18]$$

$$\dot{\varphi}+(\dot{\alpha}+\dot{\psi})\cos\theta = const \quad [9.19]$$

Furthermore, all the conditions are fulfilled to obtain the energy integral [9.11]: $E^c + \mathcal{V} = const$ (in fact, there is no complementary constraint equation here). Using expressions [8.21] and [8.22] for the kinetic energy and the potential, the energy integral gives

$$\dot{\theta}^2 + (\dot{\alpha}+\dot{\psi})^2\sin^2\theta + \left[\dot{\varphi}+(\dot{\alpha}+\dot{\psi})\cos\theta\right]^2 + 2\dot{\alpha}^2 + 2\dot{\alpha}(\dot{\alpha}+\dot{\psi})\sin\theta\sin\psi - 2\dot{\alpha}\dot{\theta}\cos\theta\cos\psi + \frac{mg}{a}\cos\theta = const \quad [9.20]$$

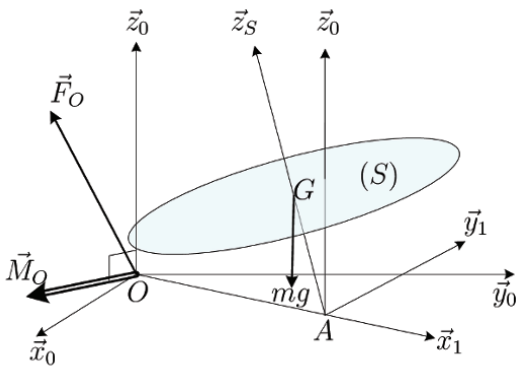


Figure 9.2. *Rigid body connected to a rotating rod by a spherical joint*

9.5.1. *First integrals via Newtonian mechanics*

The first integrals found above may also be obtained using Newtonian mechanics.

• Since $\mathcal{M}_{ext\to S}(O\vec{z}_0) = 0$ (see Figure 9.2) where the axis $O\vec{z}_0$ is fixed, we have the angular momentum first integral $\sigma_{0S}(O\vec{z}_0) = const$.

With the angular momentum $\sigma_{0S}(O\vec{z}_0)$ obtained in [8.31], we do indeed arrive again at the first integral [9.18].

• Furthermore, the moment of external efforts exerted on the rigid body S_1 about axis $C\vec{z}_S$ is zero, where C is the center of mass of S_1, $\vec{z}_S$ is attached to S_1, the inertia operator in C is axisymmetric about the axis $C\vec{z}_S$ and the moment of inertia of S_1 about the same axis is non-zero. All this leads to Euler's first integral:

$$\dot{\varphi} + (\dot{\alpha} + \dot{\psi})\cos\theta = const$$

which is the same as the first integral [9.19].

• Finally, as the efforts exerted on S do no power (perfect joint at A) or are derivable from a potential energy E^p (here $E^p = \mathcal{V}$), we also have the energy integral: $E^c + E^p = const$: this is [9.20].

9.6. Example: rigid body connected to a rotating rod by a spherical joint (no. 2)

Consider the mechanical system in section 8.5 where the rod OA is constrained to rotate at a constant velocity ω.

Equation [8.30] immediately gives the first integral:

$$\dot{\varphi} + (\omega + \dot{\psi})\cos\theta = const$$

Using the total parameterization from section 8.5.1, we do not have a Painlevé's first integral because the complementary constraint equation $\dot{\alpha} = \omega$ is not homogeneous.

However, let us prove that there exists a Painlevé's first integral using the independent parameterization from section 8.5.2. The hypotheses for Painlevé's first integral [9.6] are satisfied. Indeed:

– The spherical joint at point A between the rigid body S_1 and the rod OA is perfect.

– Recall expression [8.32] for the kinetic energy obtained using the independent parameterization:

$$E^c_{0S} = ma^2 \left\{ \dot{\theta}^2 + (\omega + \dot{\psi})^2 \sin^2\theta + \left[\dot{\varphi} + (\omega + \dot{\psi})\cos\theta\right]^2 + 2\omega^2 + \right.$$
$$\left. 2\omega(\omega + \dot{\psi})\sin\theta\sin\psi - 2\omega\dot{\theta}\cos\theta\cos\psi \right\}$$

as well as expression [8.22] for the potential due to the weight of the system:

$$\mathcal{V} = mga\cos\theta + const$$

Thus, the hypothesis $\dfrac{\partial}{\partial t}(E^c - \mathcal{V}) = 0$ is satisfied.

– There is no complementary constraint equation.

From the previous expression for the kinetic energy, we can derive

$$E^{c(2)} = ma^2 \left[\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta + (\dot{\varphi} + \dot{\psi}\cos\theta)^2\right]$$
$$E^{c(0)} = ma^2\omega^2(3 + 2\sin\theta\sin\psi)$$

Hence, Painlevé's first integral is

$$ma^2 \left[\dot{\theta}^2 + \dot{\psi}^2 \sin^2\theta + (\dot{\varphi} + \dot{\psi}\cos\theta)^2 - \omega^2(3 + 2\sin\theta\sin\psi)\right] - mga\cos\theta = const$$

9.7. Example: rigid body subjected to a double contact

Consider the rigid body subjected to a double contact, which is studied in section 8.6 with the independent parameterization.

The Lagrange's equation [8.46] immediately gives a first integral:

$$\left[6\,(k - \cos\theta)^2 + \sin^2\theta\right]\dot{\psi} - 2k\omega\,(k - \cos\theta) = const \qquad [9.21]$$

Let us show that there exists a Painlevés first integral. In order to do this, let us verify that the hypotheses for Painlevé's first integral [9.6] are satisfied:

– The contacts at points I and K between the rigid body S_1 and the exterior are perfect joints.

– The kinetic energy and the potential of the rigid body S_1 with respect to R_0 are obtained in [8.43] and [8.44], respectively:

$$E^c_{01} = \frac{1}{2}ma^2 \left\{5\dot{\theta}^2 + \left[6(k - \cos\theta)^2 + \sin^2\theta\right]\dot{\psi}^2 - 4k\omega(k - \cos\theta)\dot{\psi} + 2k^2\omega^2\right\}$$

$$\mathcal{V} = mgz + const = 2mga\sin\theta + const$$

Thus, the hypothesis $\dfrac{\partial}{\partial t}(E^c - \mathcal{V}) = 0$ is satisfied.

– Finally, there is no complementary constraint equation.

Painlevé's equation [9.6] is written as

$$5\dot{\theta}^2 + \left[6(k-\cos\theta)^2 + \sin^2\theta\right]\dot{\psi}^2 - 2k^2\omega^2 + 4\frac{g}{a}\sin\theta = const \qquad [9.22]$$

The two first integrals [9.21] and [9.22] constitute two equations for the two unknowns ψ and θ, which are simpler to solve than the Lagrange's equations [8.45] and [8.46].

9.7.1. *Using Newtonian mechanics to find a first integral*

We propose arriving at the first integral [9.21] using Newtonian mechanics. The forces exerted on the rigid body (S_1) are:

– the force $\vec{F}_K$ exerted by the axis $O\vec{z}_0$ at point K. It is parallel to $\vec{n}$ because it is orthogonal to both $\vec{z}_0$ and $\perp\ \vec{z}_1$;

– the force $\vec{F}_I$ exerted by the plate at point I, which is resolved in the basis $(\vec{n}, \vec{u}, \vec{z}_0)$ as $\vec{F}_I = X\vec{n} + Y\vec{u} + Z\vec{z}_0$;

– the weight of the system, applied at center C.

By using $\sigma_{01}(O\vec{z}_0)$ and $\delta_{01}(O\vec{z}_0)$ to denote the angular momentum and the dynamic moment, with respect to R_0, about the axis $O\vec{z}_0$ of the rigid body S_1 (see [1.76] for the definition of the dynamic moment), we have the following implications:

$$\left.\begin{array}{r}\mathcal{M}_{ext\to1}(O\vec{z}_0) = OI\ X = 2akX \\ \mathcal{M}_{ext\to1}(C\vec{z}_1) = 2aX\end{array}\right\} \quad \begin{array}{l}\to \quad \mathcal{M}_{ext\to1}(O\vec{z}_0) - k\mathcal{M}_{ext\to1}(C\vec{z}_1) = 0 \\ \to \quad \delta_{01}(O\vec{z}_0) - k\delta_{01}(C\vec{z}_1) = 0 \\ \to \quad \dfrac{d}{dt}\sigma_{01}(O\vec{z}_0) - k\underbrace{2ma^2}_{C}\dfrac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta) = 0\end{array}$$

We thus arrive at the first integral:

$$\sigma_{01}(O\vec{z}_0) - 2kma^2(\dot{\varphi} + \dot{\psi}\cos\theta) = const \qquad [9.23]$$

We obtain the explicit expression for the angular momentum $\sigma_{01}(O\vec{z}_0)$ in the previous relationships as follows:

$$\begin{aligned}\sigma_{01}(O\vec{z}_0) &= \sigma_{01}(C\vec{z}_1) + m(\overrightarrow{OG}, \vec{V}_{01}(C), \vec{z}_0) \\ &= ma^2\left[\dot{\psi}\sin^2\theta + 2(\dot{\varphi} + \dot{\psi}\cos\theta)\cos\theta\right] + 4ma^2(k-\cos\theta)^2\dot{\psi} \ \text{ where } \dot{\varphi} = k(\omega - \dot{\psi}) \\ &= ma^2\left[\left(4(k-\cos\theta)^2 + \sin^2\theta\right)\dot{\psi} + 2(\cos\theta - k)\dot{\psi}\cos\theta + 2k\omega\cos\theta\right]\end{aligned}$$

On the other hand, taking into account the constraint equation [8.40], $\varphi = k(\omega t - \psi)$, the second term of [9.23] can be written as

$$2kma^2(\dot{\varphi} + \dot{\psi}\cos\theta) = 2kma^2\left[-(k-\cos\theta)\dot{\psi} + k\omega\right]$$

By inserting these results in [9.23], we once again arrive at the first integral [9.21].

10

Equilibrium

This chapter is devoted to equilibrium, which is a particular case of motion.

• We will work in the Galilean reference frame R_g and we will adopt the same convention [6.1] as used for the Lagrange's equations, namely, we choose the common reference frame R_0 equal to R_g:

$$\boxed{R_0 = R_g}$$

• NOTATION CONVENTION: To simplify the writing in this chapter, the kinetic energy and the potential will be written without subscript: E^c, $\mathcal{V}$, instead of $E^c_{R_g\mathcal{S}}$, $\mathcal{V}_{R_g}$.

10.1. Definitions

10.1.1. *Absolute equilibrium*

Consider a system $\mathcal{S}$ made up of one or more rigid bodies.

Definition. The system $\mathcal{S}$ is in *absolute equilibrium* (implicitly: with respect to, or relative to, the Galilean reference frame R_g) if it is at rest with respect to R_g, that is, if each particle of the system has the same position in R_g over time.

The constant position occupied by $\mathcal{S}$ is called *an absolute equilibrium position*. [10.1]

Thus, if the system $\mathcal{S}$ is in absolute equilibrium relative to the Galilean reference frame R_g, the position vector $\overrightarrow{O_gP^{(g)}}$ of the current particle p of the system in R_g does not depend on time:

$$\overrightarrow{O_gP^{(g)}} = \bar{\bar{Q}}_{g0}.\overrightarrow{OP} = const \text{ with respect to } t$$

where O_g is a *fixed point in* R_g and $P^{(g)}$ is the position of p in R_g. By virtue of [1.47], the velocity, with respect to R_g, of any particle of the system is zero:

$$\vec{V}_{R_g}(p,t) = \frac{d_{R_g}\overrightarrow{O_gP}}{dt} = \bar{\bar{Q}}_{0g}.\frac{d}{dt}\left(\bar{\bar{Q}}_{g0}.\overrightarrow{O_gP}\right) = \vec{0}$$

Theorem. Necessary condition for absolute equilibrium. If a system of rigid bodies $\mathcal{S}$ is in absolute equilibrium with respect to a Galilean reference frame R_g, then, in any VVF $\dot{q}^*$, the VP of the external constraint efforts applied to $\mathcal{S}$ is zero.

This result should be compared with the analogous theorem in Newtonian mechanics, resulting from Newton's laws:

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

Theorem. Necessary condition for absolute equilibrium. If a system of rigid bodies $\mathcal{S}$ is in absolute equilibrium with respect to a Galilean reference frame R_g, then the moment field of the external efforts applied to $\mathcal{S}$ is zero.

10.1.2. *Parametric equilibrium*

We will introduce a new concept of equilibrium, slightly different from absolute equilibrium and related to the position parameters of the system. The existing mechanical joints in the system $\mathcal{S}$ are expressed by a certain number of constraint equations, which may be classified as either primitive or complementary. The retained parameters are $q \equiv (q_1, \ldots, q_n)$ and t.

Definition. [10.2]

The system $\mathcal{S}$ is in *parametric equilibrium* if the position parameters q take a *constant* value q_e over time; in other words, if the motion $t \mapsto q(t) = q_e = const$ is *one* solution to the problem whose initial conditions are, necessarily, $(t_0, q_0 = q_e, \dot{q}_0 = 0)$.

The position of the system $\mathcal{S}$ defined by $q_e = const$ is called *a parametric equilibrium position*. The term "parametric" will often be implied, so that the term *equilibrium* carries the meaning *parametric equilibrium*.

Using a contracted form, we also say that $q_e = const$ is a (parametric) equilibrium position.

As the position of the system is defined by q and t, the fact that q is constant does *not* mean that the system is at rest with respect to R_g. A parametric equilibrium is *not* necessarily an absolute equilibrium.

EXAMPLE. Consider, for example, the reference frame R_1 rotating, with respect to R_g, about the axis $O\vec{z}_g$ at the angle αt. Let $(O; \vec{x}_g, \vec{y}_g, \vec{z}_g)$ and $(O; \vec{x}_1, \vec{y}_1, \vec{z}_1)$ (with $\vec{z}_1 = \vec{z}_g$) be the coordinate systems attached to R_g and R_1 respectively (Figure 10.1). Consider the system made up of a single particle p and assume that the position of p in R_g is defined by four parameters: the three coordinates x, y, z of P in the rotating coordinate system $(O; \vec{x}_1, \vec{y}_1, \vec{z}_1)$, and the time t via the rotation angle αt.

We have a parametric equilibrium if the parameters x, y, z are constant over time, that is, if the position of the particle p in the rotating reference frame R_1 is fixed. This does not, however, mean that the particle is at rest in R_g. ■

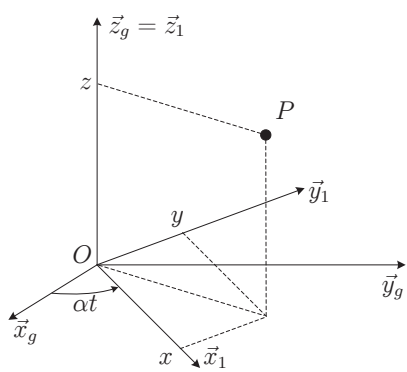


Figure 10.1. *Example of parametric equilibrium*

Consider a reference frame R_1 whose motion relative to R_g (*the background motion*, see definition [1.70]) is known and assume that the position of the system $\mathcal{S}$ in R_1 is defined by the

parameters q, not by time. Then, a parametric equilibrium position of $\mathcal{S}$ is *a relative equilibrium position* in R_1.

The following theorem is a case of a simple situation:

Theorem. If the position P of the current particle of the system satisfies $\dfrac{\partial_{R_g}\overrightarrow{O_gP}}{\partial t} = \vec{0}$ (see the definition of this derivative in [1.41]), then the concepts of absolute equilibrium and parametric equilibrium are *identical*.

PROOF. Taking into account the hypothesis, relationship [2.25] gives

$$\vec{V}_{R_g}(p,t) = \bar{\bar{Q}}_{0g}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{g0}.\overrightarrow{O_gP}\right)\dot{q}_i + \frac{\partial_{R_g}\overrightarrow{O_gP}}{\partial t} = \bar{\bar{Q}}_{0g}.\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{g0}.\overrightarrow{O_gP}\right)\dot{q}_i$$

Consequently:

– Consider a parametric equilibrium position defined by $q = q_e = const$, then the previous relationship gives $\vec{V}_{R_g}(p,t) = \vec{0}$ for any particle p of the system at any instant: this is an absolute equilibrium.

– Reciprocally, consider an absolute equilibrium, $\vec{V}_{R_g}(p,t) = \vec{0}$, then, because $\bar{\bar{Q}}_{0g}$ is invertible:

$$\sum_{i=1}^{n}\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{g0}.\overrightarrow{O_gP}\right)\dot{q}_i = \vec{0}$$

By accepting that the vectors $\frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{g0}.\overrightarrow{O_gP}\right)$ are independent, we derive that $\dot{q}_i = 0, \forall i \in [1,n]$. ■

Let us indicate that in mechanics, we also encounter the concept of steady motion, which is close to the concept of parametric equilibrium but which will not be studied here:

Definition. *A steady motion* is a motion in which:

– certain parameters remain constant,

– while the velocities of other parameters also remain constant.

10.2. Equilibrium equations

It has been seen in section 6.3 that in the general framework of dynamics, we have $n + n_{f\ell}$ unknowns (n kinematic unknowns q and $n_{f\ell}$ effort unknowns) and $n + n_{f\ell}$ equations [6.7] to solve:

$$\begin{cases} \bullet\ n \text{ Lagrange's equations,} \\ \bullet\ \ell \text{ complementary constraint equations,} \\ \bullet\ n_{f\ell} - \ell \text{ contact laws.} \end{cases}$$

Here, in an equilibrium problem, according to definition [10.2], we must make $q(t)$ equal to q_e in all these equations.

Definition. *An equilibrium equation* is a Lagrange's equation written at an equilibrium position q_e, in other words, a Lagrange's equation in which we replace $q(t)$ by q_e. [10.3]

As concerns the possible complementary constraint equations, it can be seen that:

– when we make $q = q_e$ in *holonomic* relationships, these are written as $f(q_e, t) = 0$, which implies that *the holonomic constraint equations, if they exist, must not explicitly depend on time*. They must, thus, be of the form

$$\boxed{f(q) = 0}$$

– when we make $(q, \dot{q}) = (q_e, 0)$ in *non-holonomic* relationships of the differential form [2.12], these give $\beta(q_e, t) = 0$: the non-holonomic, complementary constraint equations, if they exist, must be *homogeneous*, thus they must be of the form

$$\boxed{\sum_{i=1}^{n} \alpha_i(q, t)\, \dot{q}_i = 0}$$

Thus, at equilibrium, these constraint equations become the identities $\sum_{i=1}^{n} \alpha_i(q, t).\ 0 = 0$ and *give no equations*.

To conclude, in an equilibrium problem, we have $n + n_{f\ell}$ unknowns (n cinematic unknowns $q_e = const$ and $n_{f\ell}$ unknown constraint efforts) and $n + n_{f\ell}$ equations to be solved:

$$\begin{cases} \bullet \text{ the } n \text{ equilibrium equations,} \\ \bullet \text{ the } \textit{holonomic}\text{, complementary constraint equations written at equilibrium: } \boxed{f(q_e) = 0} \\ \bullet \text{ the } n_{f\ell} - \ell \text{ contacts laws, to be made explicit on a case-by-case basis.} \end{cases} \qquad [10.4]$$

The parametric equilibrium positions $q(t) = q_e = const$ with respect to R_g are the *constant* solutions of this system of equations.

We will list the equations and unknowns later in this section, after having obtained an explicit expression for the equilibrium equations.

In the case of an *independent* parameterization, there is no constraint equations and we have only to solve the equilibrium equations and any possible contact laws.

Finally, we may have to verify, at the equilibrium positions, *the inequalities* imposed by unilateral joints. For example, in the case of a point contact, the normal contact force N must satisfy $N \geq 0$; if we assume no-slip contact, another inequality must be verified, of the type $\|\vec{T}\| \leq fN$, where $\vec{T}$ is the tangential contact force and f is the coefficient of friction.

• Let us now establish an explicit expression for the equilibrium equations. In order to do this, let us first recall expression [2.55] for the parameterized kinetic energy $E^c(q, \dot{q}, t)$:

$$2E^c(q, \dot{q}, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(q, t)\, \dot{q}_i \dot{q}_j + 2 \sum_{i=1}^{n} b_i(q, t)\, \dot{q}_i + c(q, t) \qquad [10.5]$$

where, since convention [6.1], which automatically satisfies hypothesis [2.33], is adopted, the coefficients a_{ij}, b_i and c simplify a little here, in comparison with [2.56]:

$$a_{ij} = \int_S \frac{\overrightarrow{\partial P}}{\partial q_i} . \frac{\overrightarrow{\partial P}}{\partial q_j} dm \qquad b_i = \int_S \frac{\overrightarrow{\partial P}}{\partial q_i} . \frac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} dm \qquad c = \int_S \left(\frac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} \right)^2 dm \geq 0$$

The parameterized kinetic energy $E^c_{R_1S}(q,\dot{q},t)$ is decomposed according to [2.57]:

$$E^c = E^{c(2)} + E^{c(1)} + E^{c(0)} \qquad \text{with } E^{c(0)}(q,t)) \equiv \frac{1}{2}c(q,t) \geq 0$$

Furthermore, we divide the given efforts into two categories, as usual:

– those derivable from a potential $\mathcal{V}$, which yield the generalized forces $-\dfrac{\partial \mathcal{V}}{\partial q_i}, i \in [1,n]$,

– those which are not derivable from a potential, which yield the generalized forces denoted by $D'_i, i \in [1,n]$.

This makes it possible to express the generalized forces D_i associated with given efforts and the generalized forces Q_i in the form

$$D_i = -\frac{\partial \mathcal{V}}{\partial q_i} + D'_i \qquad \text{and} \qquad Q_i = -\frac{\partial \mathcal{V}}{\partial q_i} + D'_i + L_i$$

Theorem.

HYPOTHESIS: The reference frame R_g is *Galilean* and we choose $R_0 = R_g$ according to convention [6.1].

Then, *the equilibrium equations* are written as

$$\forall t, \forall i \in [1,n], \qquad \boxed{\frac{\partial b_i}{\partial t}_{|q_e} - \frac{\partial E^{c(0)}}{\partial q_i}_{|q_e} + \frac{\partial \mathcal{V}}{\partial q_i}_{|q_e} = D'_i(q_e,0,t) + L_i} \qquad [10.6]$$

PROOF. Recall the Lagrange's equations [6.5]:

$$\forall i \in [1,n], \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E^c}{\partial \dot{q}_i} - \frac{\partial E^c}{\partial q_i} + \frac{\partial \mathcal{V}}{\partial q_i} = D'_i + L_i \qquad [10.7]$$

A parametric equilibrium position $q(t) = q_e = const$ satisfies the same equations obtained by replacing $(t,q,\dot{q},\ddot{q})$ with $(t,q_e,0,0)$. On account of [10.5], it is possible to get explicit expressions for the derivatives of kinetic energy in [10.7]. After simple but tedious calculations, we obtain

$$\forall i \in [1,n], \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial E^c}{\partial \dot{q}_i} - \frac{\partial E^c}{\partial q_i} = \sum_j a_{ij}\ddot{q}_j + \sum_{j,k}\left(\frac{\partial a_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial a_{jk}}{\partial q_i}\right)\dot{q}_j\dot{q}_k + \sum_j \frac{\partial a_{ij}}{\partial t}\dot{q}_j$$
$$+ \sum_j \left(\frac{\partial b_i}{\partial q_j} - \frac{\partial b_j}{\partial q_i}\right)\dot{q}_j + \frac{\partial b_i}{\partial t} - \frac{\partial E^{c(0)}}{\partial q_i}$$

By making $(q,\dot{q},\ddot{q}) = (q_e,0,0)$, we obtain the desired result. Q.E.D. ■

To better understand how the equilibrium equations are obtained and to better understand their nature, let us write the Lagrange's equations [10.7] in shortened form:

$$\Phi_i(t,q,\dot{q},\ddot{q}) = L_i(t,q,\dot{q},\mathcal{F}_{constraint}),\ i \in [1,n],$$

where Φ_i denotes the terms other than L_i in [10.7]. In the general case, the generalized force L_i depends on $(t,q,\dot{q})$ and the constraint efforts denoted, symbolically, as $\mathcal{F}_{constraint}$.

The equilibrium equations [10.6] are obtained by making $q(t) = q_e = const$:

$$\forall i \in [1,n], \qquad \boxed{\Phi_i(t,q_e,0,0) = L_i(t,q_e,0,\mathcal{F}_{constraint|q_e})}$$

10.2.1. *List of equations and unknowns*

Let us assume that there are ℓ complementary constraint equations, made up of ℓ_1 holonomic equations and of ℓ_2 non-holonomic equations ($\ell = \ell_1 + \ell_2$).

– There are n kinematic unknowns q_e, and $n_{f\ell}$ constraint efforts that appear in the expressions for the generalized forces L_i. That is, $n + n_{f\ell}$ unknowns in total.

– Consider the set of equations [10.4] to be solved. The equilibrium equations [10.6] and the equations $f(q_e) = 0$ give $n + \ell_1$ equations. It has been seen that at equilibrium the non-holonomic, complementary constraint equations do not yield any equations.

  According to the discussion in section 6.3, we know that the contact laws normally give $n_{f\ell} - \ell \geq 0$ equations. Thus, we have a total of $n + n_{f\ell} - \ell_2$ equations.

In general, there are more unknowns than equations and the found solution depends on ℓ_2 arbitrary constants and we obtain *ranges of equilibrium positions*, rather than a finite number of equilibrium positions. An example of this is given in section 10.9.

In the case where there is no non-holonomic complementary constraint equation ($\ell_2 = 0$) (this is, in particular, the case when the parameterization is *independent*), we have as many equations as unknowns.

We will see, further on, that in the different cases of perfect joints we get more information on the number $n_{f\ell}$ of constraint efforts.

Existence and uniqueness of the solution
The problem of the existence and uniqueness of the solution has been discussed in section 6.4 in the general dynamic framework. It is formulated as follows for an equilibrium problem:

Given an initial instant t_0 and the initial condition $q_0 = q_e$, is there a solution $t \mapsto q(t) = q_e$ to equations [10.4] that satisfies $q(t_0) = q_e$ and $\dot{q}(t_0) = 0$? If yes, then is this solution unique?

We saw, in section 6.4, that when the solution exists, it is often unique if there are no inequalities to satisfy in addition to equations [10.4]. Assuming the existence and uniqueness of the solution, we have the following result, which justifies seeking solutions to the system of equations [10.4]:

Theorem.
HYPOTHESES:

i) The solution exists and is unique.

ii) q_e satisfies equations [10.4].

iii) At the initial instant t_0, the system $\mathcal{S}$ is released from rest at position q_e.

Then, $\forall t \geq t_0$, the system $\mathcal{S}$ remains at equilibrium with respect to R_g.

PROOF. The equilibrium $t \mapsto q(t) = q_e$ is a possible motion and is the only possible motion by virtue of the uniqueness hypothesis (i). ■

10.2.2. *The explicit presence of time in equilibrium equations*

The system of equations [10.4] does not necessarily accept constant solutions q_e:

– Very often, this system does not explicitly contain the time variable t and we find constant solutions q_e.

– However, in certain cases, t appears explicitly in the equations, for example, when the given forces are dependent on time (and possibly on $q, \dot{q}$). The system [10.4] then may not have solutions or may have solutions that are not constant.

When trying to find equilibrium, if t appears explicitly in the equations, we must retain only constant solutions from the existing solutions.

EXAMPLE. Consider a mechanical system with two retained parameters $q = (q_1, q_2)$ and assume that equations [10.4] are reduced to two equations of motion:

$$\begin{cases} tq_1 - q_2 = 0 \\ q_1 - t^2 q_2 = 0 \end{cases}$$

We find a single solution, $q_1 = q_2 = 0$, which is a constant solution and which is, therefore, an equilibrium position.

On the other hand, if equations [10.4] are

$$\begin{cases} tq_1 - q_2 = 0 \\ 1 - t^2 q_2 = 0 \end{cases}$$

we then find a single solution, $q_2 = tq_1 = 1/t^2$, and there is no equilibrium position. ■

The previous example shows that we may have equilibrium even if time appears explicitly in the equations in the problem. Having said this, the case where time is explicitly involved in the equations remains exceptional.

10.3. Equilibrium equations in the case of perfect joints and independent parameterization

We will study the case of *perfect* joints where more specified expressions for the equilibrium equations can be found, and we will start with an *independent* parameterization.

Theorem.

HYPOTHESES:

(i) The reference frame R_g is Galilean and we choose $R_0 = R_g$ according to Convention [6.1].

(ii) All the joints are *perfect*.

(iii) There is *no* complementary constraint equation (i.e. the parameterization is *independent*).

Then, *the equilibrium equations* are written as

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{\partial b_i}{\partial t}_{|q_e} - \frac{\partial E^{c(0)}}{\partial q_i}_{|q_e} + \frac{\partial \mathcal{V}}{\partial q_i}_{|q_e} = D'_i(q_e, 0, t)} \qquad [10.8]$$

PROOF. Under the above-mentioned hypotheses, the Lagrange's equations are [8.2]:

$$\forall t, \forall i \in [1, n], \qquad \frac{d}{dt}\frac{\partial E^c}{\partial \dot{q}_i} - \frac{\partial E^c}{\partial q_i} + \frac{\partial \mathcal{V}}{\partial q_i} = D'_i$$

A parametric equilibrium position $q(t) = q_e = const$ satisfies these equations with $(t, q, \dot{q}, \ddot{q}) = (t, q_e, 0, 0)$. The explicit expressions for the derivatives of the kinetic energy are obtained as in the proof for [10.6]. ■

10.3.1. *List of equations and unknowns*

– The generalized forces D'_i due to the given efforts, which do not admit of a potential, are known as a function of (q_e, t). There is no unknown constraint efforts in the equilibrium equations [10.8].

– Consider the set of equations [10.4] to be solved. Apart from the equilibrium equations, there is no other equation to be satisfied, as there is no complementary constraint equation. The contact laws mentioned in section 10.2 are already rendered through the hypothesis of perfect joints.

We thus have as many equations as unknowns. The equilibrium equations [10.8] constitute a system of n algebraic equations with n unknowns q_e.

It turns out that among all the equilibrium cases studied in this chapter, it is only in this section that the equations/unknowns list is so clear and that finding equilibrium solutions is so simple. Thus, when the joints are perfect, we had better choose an independent parameterization, whenever possible, so as to fall into the framework in this section.

Nonetheless, it is not always possible to choose an independent parameterization: when there are non-holonomic constraint equations or when there are holonomic equations that cannot be written in resolved form, these equations must be classified as complementary. We then fall into the framework in section 10.4.

From [10.8], we can derive some common specific cases.

Corollary 1.
In addition to the hypotheses in [10.8], the following hypotheses are assumed:
HYPOTHESES:

(iv) The kinetic energy E^c does *not* explicitly depend on time t.

(v) All the given efforts are derivable from a potential $\mathcal{V}$.

Then, *the equilibrium equations* are written as

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{\partial \mathcal{V}^*}{\partial q_i}_{|q_e} \equiv \frac{\partial(\mathcal{V} - E^{c(0)})}{\partial q_i}_{|q_e} = 0} \qquad [10.9]$$

where $\mathcal{V}^* \equiv \mathcal{V} - E^{c(0)}$ is called *the modified potential*.

PROOF. Hypothesis (iv) implies that $\forall i, \dfrac{\partial b_i}{\partial t} = 0$ in [10.8]. Hypothesis (v) implies that $D'_i = 0$. ■

Equation [10.9] and hypothesis (iv) entail that it must be assumed, consistently, that the potential $\mathcal{V}$ is time independent: $\mathcal{V} = \mathcal{V}(q)$.

Note in passing that if we adopt the hypotheses for [10.9] and the hypothesis $\mathcal{V} = \mathcal{V}(q)$, we obtain Painlevé's first integral [9.6]: $E^{c(2)} - E^{c(0)} + \mathcal{V} = const$.

Corollary 2.
In addition to the hypotheses from [10.8], the following hypotheses are assumed:
HYPOTHESES:

(iv)$_b$ $\dfrac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} = \vec{0}$, that is, the system does not undergo any background motion. (This hypothesis is different from hypothesis (iv) above.)

(v) All the given efforts are derivable from a potential $\mathcal{V}$.

Then, *the equilibrium equations* can be written as

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{\partial \mathcal{V}}{\partial q_i}_{|q_e} = 0} \qquad [10.10]$$

The equilibrium positions are *stationary points* of the potential.

PROOF. If $\dfrac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} = \vec{0}$, then $\forall i, b_i = 0$ and $E^{c(0)} = 0$ in [10.8]. Hypothesis (v) implies that $D'_i = 0$. ■

10.4. Equilibrium equations in the case of perfect joints and in the presence of complementary constraint equations

Let us now examine the case where all the joints are perfect and where the chosen parameterization includes complementary constraint equations.

Theorem :

HYPOTHESES:

(i) The reference frame R_g is Galilean and we choose $R_0 = R_g$ according to convention [6.1].

(ii) All the joints are *perfect*.

(iii) There exist ℓ ($\ell < n$) (independent) *complementary* constraint equations made up of:

- ℓ_1 *holonomic* equations of the form $f(q) = 0$, which do *not* explicitly depend on time,
- ℓ_2 *non-holonomic* equations in *homogeneous* differential form $\displaystyle\sum_{i=1}^{n} \alpha_i(q, t)\ \dot{q}_i = 0$.

Thus, *the equilibrium equations* are written as

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{\partial b_i}{\partial t}_{|q_e} - \frac{\partial E^{c(0)}}{\partial q_i}_{|q_e} + \frac{\partial \mathcal{V}}{\partial q_i}_{|q_e} = D'_i(q_e, 0, t) + \sum_{h=1}^{\ell=\ell_1+\ell_2} \lambda_h\ \alpha_{hi}(q_e, t)} \qquad [10.11]$$

In the above expression:

- To write the sum $\displaystyle\sum_{h=1}^{\ell=\ell_1+\ell_2} \lambda_h\ \alpha_{hi}(q_e, t)$, we recast the ℓ_1 holonomic constraint equations into differential forms as with the ℓ_2 non-holonomic equations, in such a way as to have $\ell = \ell_1 + \ell_2$ constraint equations written in a differential form $\displaystyle\sum_{i=1}^{n} \alpha_{hi}(q, t)\ \dot{q}_i = 0$, $h \in [1, \ell = \ell_1 + \ell_2]$.
- The Lagrange multipliers λ_h are unknown scalars.

PROOF. Under the above-mentioned hypotheses, any motion $t \mapsto q(t)$ satisfies the Lagrange's

equations [8.6]:

$$\forall t, \forall i \in [1, n], \qquad \frac{d}{dt}\frac{\partial E^c}{\partial \dot{q}_i} - \frac{\partial E^c}{\partial q_i} + \frac{\partial \mathcal{V}}{\partial q_i} = D'_i + \sum_{h=1}^{\ell} \lambda_h \, \alpha_{hi} \qquad [10.12]$$

and the ℓ complementary constraint equations. In particular, a parametric equilibrium position $q(t) = q_e = const$ satisfies the same equations where $(t, q, \dot{q}, \ddot{q})$ is replaced with $(t, q_e, 0, 0)$.

The explicit expressions for the derivatives of the kinetic energy in [10.12] are obtained as in the proof for [10.6] and we obtain the desired result. Q.E.D. ■

10.4.1. *List of equations and unknowns*

– We have n unknowns q_e and $\ell = \ell_1 + \ell_2$ multipliers λ_h, that is, a total of $n + \ell_1 + \ell_2$ unknowns.

– Consider the set of equations [10.4] to be solved. The equilibrium equations [10.11] and the equations $f(q_e) = 0$ give only $n + \ell_1$ equations. At equilibrium, the non-holonomic complementary constraint equations do not provide any equations.

Thus, the solution found depends on ℓ_2 arbitrary constants: we obtain *ranges of equilibrium positions* rather than a finite number of equilibrium positions.

If there are no non-holonomic constraint equations ($\ell_2 = 0$), we return to the case of section 10.3 where there are as many equations as unknowns.

Finally, as in the general case in section 10.2, we must also satisfy, in these equilibrium positions, *the inequalities* imposed by the unilateral joints.

REMARK. The constraint equation was assumed to be independent. If the holonomic equations are not independent, then, even in the case where $\ell_2 = 0$ and where there are as many equations as unknowns, certain multipliers may be indeterminate (the joints are redundant and the system is hyperstatic). Assuming the independence of the constraint equations allows us to discard the case of hyperstatic systems, which serves no purpose here. ■

Corollary.

In addition to the hypotheses in theorem [10.11], the following hypotheses are assumed:

HYPOTHESES:

(iv) $\dfrac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} = \vec{0}$ (the system does not undergo any background motion).

(v) All the given efforts are derivable from a potential $\mathcal{V}$.

Then, *the equilibrium equations* are written as

$$\forall t, \forall i \in [1, n], \qquad \boxed{\frac{\partial \mathcal{V}}{\partial q_i}_{|q_e} = \sum_{h=1}^{\ell} \lambda_h \, \alpha_{hi}(q_e, t)} \qquad [10.13]$$

PROOF. If $\dfrac{\partial_{R_g} \overrightarrow{O_g P}}{\partial t} = \vec{0}$, then $\forall i, b_i = 0$, $E^{c(0)} = 0$ in [10.11]. Hypothesis (v) implies that $D'_i = 0$. ■

10.5. Stability of an equilibrium

In mechanics, it is interesting to not only know the possible (parametric) equilibria of the studied system, but also to know whether or not a given equilibrium is stable. Roughly speaking, an equilibrium q_s is *stable*, if initial conditions "close" to $(t_0, q_e, 0)$ engender a motion of the system that remains "close" to the equilibrium in question over time. In other words, the equilibrium q_e is stable if any motion corresponding to the initial conditions "close" to the conditions that yield the equilibrium remains "in the neighborhood of q_e". Conversely, we say that the equilibrium q_e is *unstable* if there are initial conditions "close" to $(t_0, q_e, 0)$ which cause the system to deviate significantly from the equilibrium.

Thus, for instance, a pendulum moving in the downward vertical gravitational field has two equilibrium positions (equilibrium instead of parametric equilibrium as there is only a single position parameter here). The equilibrium position where the center of mass of the pendulum is below the pivot is stable. This is because if at the initial instant the pendulum is slightly shifted from this position, the resulting motion is a small oscillation about this point. On the other hand, the equilibrium position where the center of mass of the pendulum is above the pivot is unstable. This is because all that is needed is a small shift from this position at the initial instant for the pendulum to deviate significantly.

The following definition is a formal, precise and rigorous, statement of the previous idea about stability.

Lyapunov stability. The parametric equilibrium position q_e is *stable* if, by definition:

$$\forall \varepsilon, \mu > 0, \exists \delta, \nu > 0, \forall \text{ motion } t \mapsto q(t) \text{ defined by the initial conditions } (t_0, q_0, \dot{q}_0)$$
$$\text{satisfying } \|q_0 - q_e\| < \delta, \|\dot{q}_0\| < \nu, \text{ we have } \forall t \geq t_0, \|q(t) - q_e\| < \varepsilon, \|\dot{q}(t)\| < \mu$$

This is stability in position and velocity, that is, $(q, \dot{q})$ remains in an arbitrarily small neighborhood of $(q_e, 0)$ (in $\mathbb{R}^{2n}$).

It was seen that in the large majority of cases, there is only a single motion defined by the initial conditions $(t_0, q_0, \dot{q}_0)$. However, the previous definition remains valid even if there is no uniqueness.

We accept the following theorem, which gives a *sufficient* condition of stability.

Lagrange–Dirichlet theorem.

HYPOTHESES: We adopt the hypotheses that lead to the equilibrium equation [10.9], yet slightly reinforcing hypothesis (v) below:

(i) The reference frame R_g is Galilean and we choose $R_0 = R_g$ in accordance with convention [6.1].

(ii) All the joints are *perfect*.

(iii) There is *no* complementary constraint equation (that is, the parameterization is *independent*).

(iv) The kinetic energy E^c does *not* explicitly depend on time t.

(v) All the given efforts are derivable from a time-independent potential $\mathcal{V}(q)$.

Under the above hypotheses, if the modified potential $\mathcal{V}^* \equiv \mathcal{V} - E^{c(0)}$ of variable q has a *strict* local minimum at q_e, then the position defined by q_e is a *stable* parametric equilibrium position.

If some parameters q_j do not appear in $\mathcal{V} - E^{c(0)}$, the minimum, if it exists, *cannot* be strict.

The Lagrange–Dirichlet theorem does not give a sufficient condition for stability. Thus, if q_e is not a strict local minimum, this theorem does not allow us to state that the equilibrium position q_e is stable. It does not either allow us to state that q_e is unstable.

The following theorem gives a *sufficient* condition for $\mathcal{V}^*$ to have a strict local minimum at q_e based on the second derivatives of $\mathcal{V}^*$ at q_e:

Theorem (Energy criterion).

HYPOTHESIS (Always satisfied in practice): $\mathcal{V}^*$, function of q alone, is of class C^2 in the neighborhood of q_e.

Let $[K]$ denote the symmetrical square matrix of order n whose (i,j)-component is $\dfrac{\partial^2 \mathcal{V}^*}{\partial q_i \partial q_j}(q_e)$.

We then have the following *implication*: $[K]$ is *positive definite*, in other words, all the eigenvalues of $[K]$ are *strictly positive* (as matrix $[K]$ is symmetric, it is diagonalizable and its eigenvalues are real)

$\Rightarrow$ $\mathcal{V}^*$ has *a strict local minimum* about q_e.

PROOF. The second-order Taylor expansion of $\mathcal{V}^*$ about q_e is written as

$$\mathcal{V}^*(q) = \mathcal{V}(q_e) + \sum_{i=1}^{n} \underbrace{\frac{\partial \mathcal{V}^*}{\partial q_i}(q_e)}_{=0}(q_i - q_{ie}) + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 \mathcal{V}^*}{\partial q_i \partial q_j}(q_e)(q_i - q_{ie})(q_j - q_{je}) + o(\|q - q_e\|^n)$$

or, in shortened form, denoting $\{q\}$ the column-vector containing the components $(q_1, \cdots, q_n)$:

$$\mathcal{V}^*(q) = \mathcal{V}(q_e) + \frac{1}{2}\{q - q_e\}^T [K] \{q - q_e\} + o(\|q - q_e\|^n) \qquad \blacksquare$$

In the particular case where there are two retained parameters, $q = (q_1, q_2)$, the matrix $[K]$ is written as

$$[K] = \begin{bmatrix} r & s \\ s & t \end{bmatrix} \qquad \text{with } r = \frac{\partial^2 \mathcal{V}^*}{\partial q_1^2}(q_e) \quad s = \frac{\partial^2 \mathcal{V}^*}{\partial q_1 \partial q_2}(q_e) \quad t = \frac{\partial^2 \mathcal{V}^*}{\partial q_2^2}(q_e)$$

If $r > 0$ and $s^2 - rt < 0$, then the equilibrium q_e is stable.

10.6. Example: equilibrium of a jack

Consider a lifting system similar to a car jack and remaining in the vertical plane $O\vec{x}\vec{y}$ of a Galilean reference frame $R_g = R_0$ (Figure 10.2). It is made up of:

- two rods AE and BD of mass $2m$ and length 2ℓ,
- two rods DH and EH of mass m and length ℓ and whose center of mass is G_1 and G_2,
- a mass m placed at H.

The rods are pinned at their ends. The point A coincides with the origin O. The rods AE and BD are pinned at their midpoints C. All these joints are perfect. An actuator, not shown in the figure, exerts a given force $\vec{F} = -F\vec{x}$ at B.

We wish to determine the equilibrium position of the jack as a function of the applied force F. The chosen parameterization is given below:

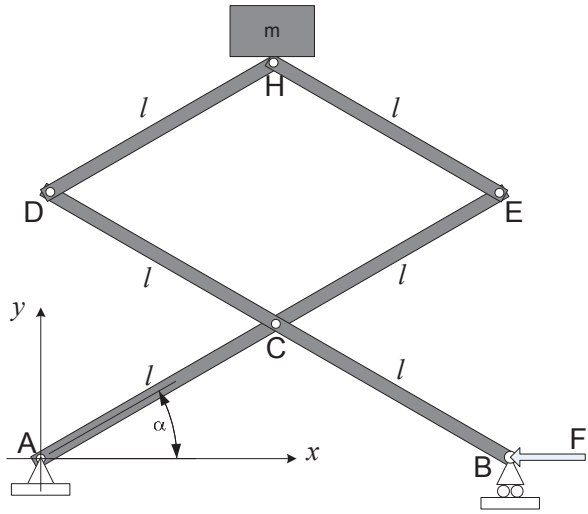


Figure 10.2. *Equilibrium of a jack*

INDEPENDENT PARAMETERIZATION.

- Primitive parameter: The angle α shown in Figure 10.2. For physical reasons, α lies between 0 and $\pi/2$.
- No primitive constraint equation.
- Retained parameter: α.
- No complementary constraint equation.

All the requirements are met so as to apply the equilibrium equation [10.10], which is written as

$$\frac{\partial \mathcal{V}}{\partial \alpha}_{|\alpha_e} = 0$$

The potential of the system is calculated using [5.22] and [5.23]:

$$\begin{aligned}\mathcal{V} &= F\vec{y}\cdot\overrightarrow{OB} + 4mg\vec{y}\cdot\overrightarrow{OC} + mg\vec{y}\cdot\overrightarrow{OG_1} + mg\vec{y}\cdot\overrightarrow{OG_2} + mg\vec{y}\cdot\overrightarrow{OH} + const\\ &= 2F\ell\cos\alpha + 12mg\ell\sin\alpha + const\end{aligned}$$

The equilibrium equation thus gives

$$-2F\ell\sin\alpha_e + 12mg\ell\cos\alpha_e = 0 \qquad \Leftrightarrow \qquad \tan\alpha_e = \frac{6mg}{F}$$

10.7. Example: equilibrium of a lifting platform

Consider a Galilean reference frame $R_g = R_0$, endowed with an orthonormal coordinate system $(O;\vec{x},\vec{y},\vec{z})$ and a lifting platform made up of two massless parallel arms I_1J_1, I_2J_2 of length ℓ, and of a horizontal platform that carries a load of mass M. These components are pinned as shown in Figure 10.3. The problem can be reduced to a plane motion in the vertical plane $O\vec{x}\vec{y}$.

An actuator outside the system exerts a force $\vec{F}$ on the system, at the point J_2, parallel to I_1J_2. We write $\vec{F} = F\dfrac{\overrightarrow{I_1J_2}}{\|\overrightarrow{I_1J_2}\|}$.

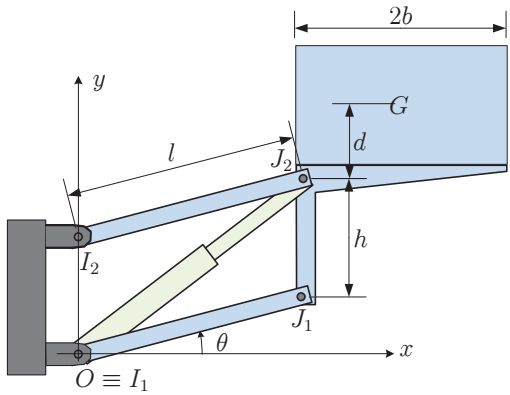


Figure 10.3. *Equilibrium of a lifting platform*

We wish to determine the force F developed by the actuator as a function of the position of the platform and the weigh Mg of the lifted load.

We choose the following parameterization:

INDEPENDENT PARAMETERIZATION.

- Primitive parameter: The angle θ between $\vec{x}$ and $\overrightarrow{I_1J_1}$.
- No primitive constraint equation.
- Retained parameter: θ. For physical reasons, it is assumed that $\theta \neq \pm\dfrac{\pi}{2}$.
- No complementary constraint equation.

Given the dimensions shown in Figure 10.3, the position vector of the mass center G of the lifted load is $\overrightarrow{OG} = (\ell\cos\theta + b)\vec{x} + (\ell\sin\theta + h + d)\vec{y}$. Hence, the potential due to the system's weight using [5.23] is given as:

$$\mathcal{V} = Mg\vec{y}\cdot\overrightarrow{OG} + const = Mg\ell\sin\theta + const$$

On the other hand, the force $\vec{F}$ exerted by the actuator at J_2 is directed along the axis I_1J_2 of the actuator:

$$\vec{F} = F\frac{\overrightarrow{I_1J_2}}{\|\overrightarrow{I_1J_2}\|} = F\frac{\ell\cos\theta\vec{x} + (\ell\sin\theta + h)\,\vec{y}}{\sqrt{\ell^2\cos^2\theta + (\ell\sin\theta + h)^2}}$$

The force $\vec{F}$ by the actuator is indeed a given force according to definition [3.6]. We do not know whether this force is derivable or not from a potential. In any case, as it depends on θ, we cannot apply relationship [5.22].

In these conditions, let us apply the equilibrium equation [10.8], knowing that here $b_\theta = 0$ and $E^{c(0)} = 0$:

$$\frac{\partial \mathcal{V}}{\partial \theta}_{|\theta_e} = D'_\theta(\theta_e, 0, t) \qquad [10.14]$$

where D'_θ is the generalized force due to the force $\vec{F}$.

Let us calculate the virtual power of the force $\vec{F}$: $\mathcal{P}^*(\vec{F}) = \vec{F}.\vec{V}^*_{R_g S_2}(J_2)$ where S_2 denotes the rigid body I_2J_2. The virtual velocity $\vec{V}^*_{R_g S_2}(J_2)$ is calculated by the VVF relationship [4.35]:

$$\begin{aligned}\vec{V}^*_{R_g S_2}(J_2) &= \vec{V}^*_{R_g S_2}(I_2) + \dot{\theta}^* \vec{z} \times \overrightarrow{I_2J_2} \text{ where } \overrightarrow{I_2J_2} = \ell\cos\theta\vec{x} + \ell\sin\theta\vec{y} \\ &= \ell(-\sin\theta\vec{x} + \cos\theta\vec{y})\dot{\theta}^*\end{aligned}$$

Hence

$$\mathcal{P}^*(\vec{F}) = \frac{Fh\ell\cos\theta}{\sqrt{\ell^2\cos^2\theta + (\ell\sin\theta + h)^2}}\dot{\theta}^*$$

The equilibrium equation [10.14] thus gives

$$Mg\cos\theta_e = \frac{Fh\cos\theta_e}{\sqrt{\ell^2\cos^2\theta_e + (\ell\sin\theta_e + h)^2}}$$

or, since $\theta \neq \pm\frac{\pi}{2}$:

$$F = \frac{Mg}{h}\sqrt{h^2 + \ell^2 + 2h\ell\sin\theta_e}$$

In particular, if $h = \ell$, the last relationship gives $F = Mg\ell\sqrt{2}\sqrt{1 + \sin\theta_e}$.

10.8. Example: equilibrium of a rod in a gutter

Let there be a cylindrical surface with a cross-section defined in the vertical plane $O\vec{x}\vec{y}$ of a Galilean reference system $R_g = R_0$ by a semicircle of axis $O\vec{z}$ and radius R. On this surface is placed a thin, homogeneous rod of length ℓ, mass m and center of mass G (Figure 10.4). The rod is subjected to the gravational field $-g\vec{y}$ and is constrained to remain in the plane $O\vec{x}\vec{y}$. The contacts at the points A and B between the rod and the cylindrical surface are assumed frictionless.

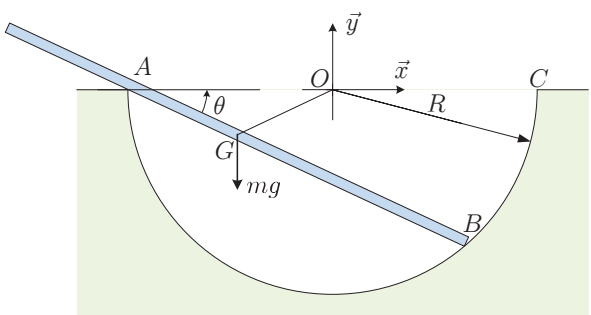


Figure 10.4. *Equilibrium of a rod in a gutter*

We choose the following parameterization:

INDEPENDENT PARAMETERIZATION.

- Primitive parameter: the angle $\theta = (\overrightarrow{AB}, \vec{x})$, measured around $\vec{z}$. Let us note that the angle θ is defined by starting from $\overrightarrow{AB}$ and not from $\vec{x}$, such that in Figure 10.4, θ is positive and lies in the interval $[0, \pi/2]$ and we have $\sin\theta \geq 0$ and $\cos\theta \geq 0$.
- No primitive constraint equation.
- Retained parameter: θ.
- No complementary constraint equation.

The hypotheses and the chosen parameterization make it possible to apply the equilibrium equation [10.10], which is written as

$$\frac{\partial \mathcal{V}}{\partial \theta}_{|\theta_e} = 0$$

The potential $\mathcal{V}$ due to the weight of the system is obtained through [5.23], $\mathcal{V} = -mg\vec{y}\cdot\overrightarrow{OG} + const$, with

$$\overrightarrow{OG} = \overrightarrow{OB} - \overrightarrow{GB} = R\left(\cos 2\theta\vec{x} - \sin 2\theta\vec{y}\right) - \frac{\ell}{2}\left(\cos\theta\vec{x} - \sin\theta\vec{y}\right)$$

Hence

$$\mathcal{V} = -mg\left(2Rcos\theta - \frac{\ell}{2}\right)\sin\theta + const$$

The equilibrium equation then gives

$$4R\cos^2\theta_e - \frac{\ell}{2}\cos\theta_e - 2R = 0$$

that is, *a priori*, two solutions for $\cos\theta_e$:

$$\cos\theta_e = \frac{1}{16}\left[\frac{\ell}{R} + \sqrt{128 + \left(\frac{\ell}{R}\right)^2}\right] > 0 \quad \text{and} \quad \cos\theta_e = \frac{1}{16}\left[\frac{\ell}{R} - \sqrt{128 + \left(\frac{\ell}{R}\right)^2}\right] < 0 \qquad [10.15]$$

We retain only the first solution, which is positive.

The equilibrium position θ_e as a function of the length ℓ of the rod is represented in Figure 10.5.

Let us verify that the contact inequalities are satisfied. With N_A, N_B denoting the contact forces at A, B, which are perpendicular to AB and to OB, respectively, these inequalities are written as

$$N_A \geq 0 \qquad \text{and} \qquad N_B \geq 0 \qquad [10.16]$$

The equilibrium of the rod gives

$$N_A = \frac{\cos 2\theta_e}{\cos\theta_e} mg \qquad\qquad N_B = \tan\theta_e\, mg$$

Knowing $\cos\theta_e$, which was found in [10.15], we can express N_A, N_B as a function of ℓ/R and derive the ℓ/R ratios that satisfy conditions [10.16].

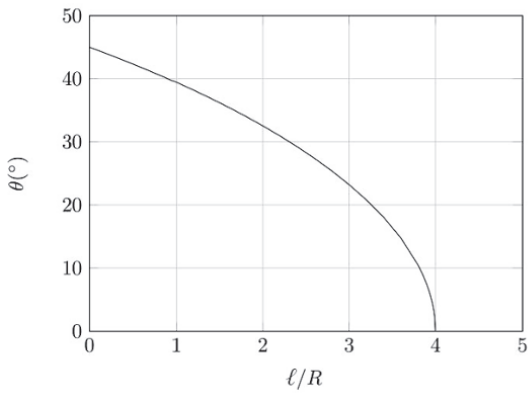


Figure 10.5. *Equilibrium position of the rod as a function of ratio ℓ/R*

The stability of the equilibrium position can be determined using the second derivative of the potential:

$$\frac{\partial^2 \mathcal{V}}{\partial \theta^2} = mg\left(8R\cos\theta + \frac{\ell}{2}\right)\sin\theta > 0 \text{ for } \theta \in [0, \pi/2]$$

The found equilibrium position θ_e is, thus, stable.

10.9. Example: existence of ranges of equilibrium positions

Consider a homogeneous disk of center C, radius R and mass m, lying in the vertical plane $O\vec{x}\vec{y}$ of a Galilean reference frame $R_g = R_0$. It is rolling without slipping over the horizontal axis $O\vec{x}$ (Figure 10.6). A load of mass M is welded at point A on the circumference of the disk. A spring of stiffness k and unstretched length L connects the center C to the fixed point $B(0, R)$.

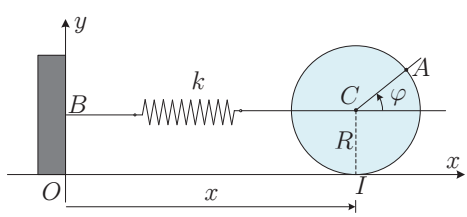


Figure 10.6. *Equilibrium of a disk connected to a spring*

As all the joints are perfect, we can use an independent parameterization and search for equilibrium positions using equation [10.8]. However, in order to verify the inequality of the no-slip condition at the contact point I, we prefer to use the following parameterization that directly gives us access to the contact efforts:

PARAMETERIZATION.

- Primitive parameters: The coordinates (x, y) of the center C, the angle φ between the vector $\vec{x}$ and the radius $\overrightarrow{AC}$ of the disk.

- No primitive constraint equation.
- Retained parameters: x, y, φ.
- Complementary constraint equations: $y = R$ and the no-slip condition $\dot{x} + R\dot{\varphi} = 0$.

Let us apply [10.6] knowing that, here, $b_i = 0$ and $E^{c(0)} = 0$:

$$\forall i \in [1,3], \qquad \frac{\partial \mathcal{V}}{\partial q_i}_{|q_e} = L_i$$

where the potential is

$$\mathcal{V} = \frac{1}{2}k(x-L)^2 + mgy + MgR\sin\varphi + const$$

The generalized forces L_i due to the constraint efforts, in this case the contact force $T\vec{x} + N\vec{y}$ at point I, are derived from the virtual power of these efforts:

$$\begin{aligned} \mathcal{P}^*_{\text{contact force}} &= (T\vec{x} + N\vec{y}).\vec{V}^*(I) \quad \text{where} \quad \vec{V}^*(I) = (\dot{x}^* + R\dot{\varphi}^*)\vec{x} + \dot{y}^*\vec{y} \\ &= T(\dot{x}^* + R\dot{\varphi}^*) + N\dot{y}^* \end{aligned}$$

The equilibrium equations thus give

$$\begin{aligned} &k(x_e - L) = T \\ &mg = N \\ &Mg\cos\varphi_e = T \end{aligned}$$

Taking into account the complementary constraint equation $y_e = R$, we have four equations for five unknowns x_e, y_e, φ_e and T, N. We find

$$x_e = \frac{Mg}{k}\cos\varphi_e + L \qquad N = mg \qquad T = Mg\cos\varphi_e$$

The angle φ_e is not completely arbitrary as there remains the no-slip condition $|T| \leq fN$ (where f is the coefficient of friction) to be satisfied:

$$|\cos\varphi_e| \leq f\frac{m}{M}$$

We thus find not a finite number, but an *infinite set of equilibrium positions (a range of equilibrium positions)*, where the abscissa x_e is expressed as a function of the angle φ_e and where this angle takes any value that satisfies the above inequality.

10.10. Example: relative equilibrium with respect to a rotating reference frame

Consider a centrifugal governor formed of several spinned arms and masses as represented in Figure 10.7. The point A is at a fixed elevation h on the vertical axis $O\vec{z}$ of a Galilean reference frame $R_g = R_0$, while the point B moves freely along the same axis. The suspension pins are at a distance a from the axis of the governor. The rods are of length b or ℓ and are massless. The joints are perfect.

The guide sleeve has the mass M located at the point B. The two centrifugal weights of mass m are each attached to the points C and D, which are symmetric with respect to the axis. The whole device rotates about the axis $O\vec{z}$ at a constant angular velocity ω.

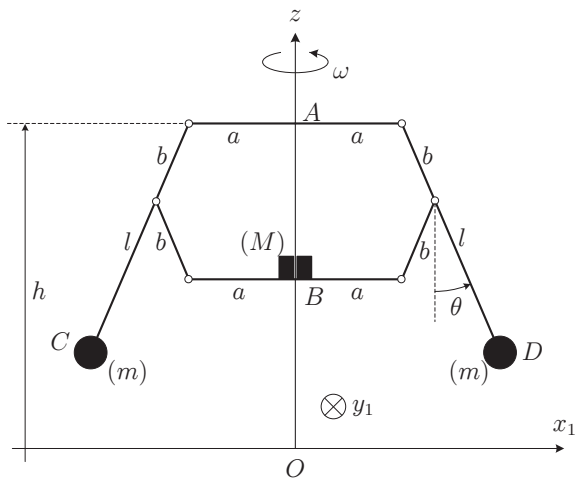


Figure 10.7. *Relative equilibrium with respect to a rotating reference frame*

The rods lie in the plane $O\vec{x}_1\vec{z}$ normal to $\vec{y}_1$; let us define the rotating reference frame R_1 by the coordinate system $(O; \vec{x}_1, \vec{y}_1, \vec{z})$ and let us study the equilibrium positions of the device relative to this reference frame.

We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: the opening angle θ between the vertical and the arms, the time t via the rotation of R_1 with respect to R_g.
- No primitive constraint equation.
- Retained parameter: θ, t.
- No complementary constraint equation.

Since $\overrightarrow{AB} = (h - 2\ell\cos\theta)\vec{z}$, the velocity of the mass at B is

$$\vec{V}(B) = \frac{d\overrightarrow{AB}}{dt} = \ell\dot{\theta}\sin\theta\vec{z}$$

Since $\overrightarrow{AD} = (a + \ell\sin\theta)\vec{x}_1 + (h - \ell\cos\theta)\vec{z}$, the velocity of the centrifugal weight at D with respect to the Galilean reference frame R_g is

$$\vec{V}(D) = \frac{d\overrightarrow{AD}}{dt} = \ell\dot{\theta}\cos\theta\vec{x}_1 + (a + \ell\sin\theta)\omega\vec{y}_1 + \ell\dot{\theta}\sin\theta\vec{z}$$

Consequently, the kinetic energy of the system (considering the mass at B to be a point mass) is

$$E^c = \frac{1}{2}M\vec{V}^2(B) + 2\frac{1}{2}m\vec{V}^2(D) = \frac{1}{2}M\ell^2\sin^2\theta\dot{\theta}^2 + m[\ell^2\dot{\theta}^2 + (a + \ell\sin\theta)^2\omega^2] \quad [10.17]$$

where the coefficient 2 before $\frac{1}{2}m$ takes into account the presence of two identical and symmetrical centrifugal weights at C and D.

The potential due to the system's weight is

$$\mathcal{V} = Mgz_B + 2mgz_D + const = -2(Mb + m\ell)g\cos\theta + const$$

The relative equilibrium positions are given by equation [10.9] since here the kinetic energy E^c does not explicitly depend on the time t and all the given efforts are derivable from a potential:

$$\frac{\partial(\mathcal{V} - E^{c(0)})}{\partial\theta}_{|\theta_e} = 0,$$

where $E^{c(0)}$ is obtained from [10.17] (refer again to definition [2.58]): $E^{c(0)} = m\omega^2(a+\ell\sin\theta)^2$. The equilibrium equation is written as

$$(Mb + m\ell)g\sin\theta_e = m\ell\omega^2(a + \ell\sin\theta_e)\cos\theta_e \qquad [10.18]$$

Solving this equation provides the equilibrium position θ_e for each given angular velocity ω. Two obvious solutions are $\theta_e = 0$ when $\omega = 0$ and $\theta_e = 90°$ when $\omega \to \infty$.

Differentiating the relationship [10.18] gives the derivative of θ_e with respect to ω:

$$\frac{d\theta_e}{d\omega} = \frac{2m\ell\omega(a + \ell\sin\theta_e)\cos\theta_e}{(Mb + m\ell)g\cos\theta_e - m\ell\omega^2[\ell\cos^2\theta_e - (a + \ell\sin\theta_e)\sin\theta_e]} \qquad [10.19]$$

For θ_e values within the interval $[0, 90°]$, the numerator on the right-hand side is positive. Let us show that this is also the case for the denominator. Because θ_e satisfies equation [10.18], we can use this equation to have

$$(Mb + m\ell)g = m\ell\omega^2(a + \ell\sin\theta_e)\frac{\cos\theta_e}{\sin\theta_e}$$

Using this result, the denominator of the right-hand side of [10.19] becomes

$$(Mb + m\ell)g\cos\theta_e - m\ell\omega^2[\ell\cos^2\theta_e - (a + \ell\sin\theta_e)\sin\theta_e] = \frac{a + \ell\sin^3\theta_e}{\sin\theta_e} > 0$$

We finally arrive at

$$\frac{d\theta_e}{d\omega} = \frac{m\ell\omega(a + \ell\sin\theta_e)\sin 2\theta_e}{a + \ell\sin^3\theta_e} > 0$$

The opening angle θ_e increases with the angular velocity ω.

10.11. Example: equilibrium in the presence of contact inequalities

Consider a massless rod AB of length a, lying in a fixed plane in a Galilean reference frame R_g. The rod is connected, at its end B, to the fixed support through a perfect pivot joint. A spring of stiffness k and unstretched length ℓ_1 connects the end A to a point C located at a distance a below B (Figure 10.8). The rotation of the road is limited by a rigid wall on the left, at a distance $d = a/2$ from BC.

We will study the equilibrium positions of the rod and their stability. The parameterization is as follows:

PARAMETERIZATION.

- Primitive parameter: the angle θ positioning the rod BA, as defined in Figure 10.8.
- No primitive constraint equation.
- Retained parameter: θ.
- No complementary constraint equation.

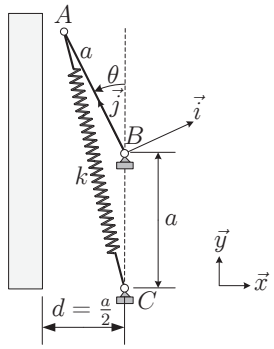


Figure 10.8. *Equilibrium of a rod connected to a spring in the presence of a wall*

Due to the presence of the wall, the angle of rotation θ of the rod cannot take arbitrary values. For the sake of convenience, let us consider that a complete rotation around B corresponds to

$$-7\pi/6 \leq \theta \leq 5\pi/6 \qquad \text{or} \qquad -210^\circ \leq \theta \leq 150^\circ$$

rather than the usual interval $0^\circ \leq \theta \leq 360^\circ$. Thus, the forbidden values of θ are $30^\circ < \theta < 150^\circ$, the permitted values are $-210^\circ \leq \theta \leq 30^\circ$ (or $-7\pi/6 \leq \theta \leq \pi/6$).

The reaction force exerted by the wall on the rod (force perpendicular to the wall, assuming that the contact between the wall and rod occurs is frictionless) is denoted by $\vec{F} = N\vec{x}$. The force N is

- non-zero if point A touches the wall, i.e. if the angle $\theta = -210^\circ$ or 30°,
- zero if $-210^\circ < \theta < 30^\circ$,
- infinite if $30^\circ < \theta < 150^\circ$ (this is a way of expressing that this interval is inaccessible).

In all the cases, N must satisfy the inequality $N \geq 0$. Accordingly, the force $\vec{F}$ can be written in the condensed form

$$\vec{F} = N\ H(\theta)\vec{x},$$

where $H(\theta)$ denotes a function similar to the Heaviside step function, equal to 1 between 30° and 150°, and zero elsewhere, as represented in Figure 10.9.

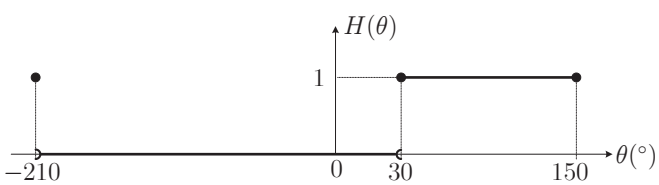


Figure 10.9. *The function $H(\theta)$ used in this example*

The equilibrium positions of the rod AB are given by equation [10.6] with $b_i = 0$ and $E^{c(0)} = 0$:

$$\frac{\partial \mathcal{V}}{\partial \theta}_{|\theta_e} = L_\theta, \qquad [10.20]$$

where L_θ is the generalized force corresponding to the reaction force $\vec{F}$. Here, the potential due to the spring AC is $\mathcal{V} = \frac{1}{2}(\ell - \ell_1)^2 + const$, where the length ℓ of the spring is

$$\ell = \sqrt{a^2 + a^2 + 2a^2 \cos\theta} = 2a|\cos\frac{\theta}{2}|$$

(the vertical bars for the absolute value are necessary as $\cos\frac{\theta}{2}$ may have negative values). Hence, with θ_1 denoting the value of θ when the spring is unstretched:

$$\mathcal{V} = 2ka^2\left(|\cos\frac{\theta}{2}| - \cos\frac{\theta_1}{2}\right)^2 + const$$

From this, we derive

$$\frac{d\mathcal{V}}{d\theta} = -2ka^2 \sin\frac{\theta}{2}\left(|\cos\frac{\theta}{2}| - \cos\frac{\theta_1}{2}\right) sgn \cos\frac{\theta}{2}$$

where sgn stands for "sign of". To obtain the generalized force L_θ, let us calculate the virtual power of the force $\vec{F}$:

$$\mathcal{P}^*_{\vec{F}} = N\ H(\theta)\vec{x}.\vec{V}^*(A)$$

where, with $\vec{j}$ denoting the unit vector in the sense of $\overrightarrow{BA}$ and $\vec{i} \equiv \vec{j} \times \vec{z}$ (see Figure 10.8):

$$\vec{V}^*(A) = \vec{V}^*(B) + \vec{\Omega}^* \times \overrightarrow{BA} = \vec{0} + \dot{\theta}^*\vec{z} \times a\vec{j} = -a\dot{\theta}^*\vec{i}$$

From this, we derive

$$L_\theta = -Na\ H(\theta)\cos\theta$$

The equilibrium equation [10.20] is thus written as

$$2ka \sin\frac{\theta_e}{2}\left(|\cos\frac{\theta_e}{2}| - \cos\frac{\theta_1}{2}\right) sgn \cos\frac{\theta_e}{2} = N\ H(\theta_e)\cos\theta_e \qquad [10.21]$$

In view of the graph for $H(\theta)$ in Figure 10.9, we will solve this equation by distinguishing between two cases: (i) $-210° < \theta_e < 30°$ and (ii) $\theta_e = -210°$ or $30°$. The interval $30° < \theta < 150°$ is not retained as N is then infinite and equation [10.21] has no solution.

1. If $-210° < \theta_e < 30°$, then $H(\theta_e) = 0$ and equation [10.21] simply becomes $\frac{\partial \mathcal{V}}{\partial \theta}_{|\theta_e} = 0$:

$$\sin\frac{\theta_e}{2}\left(|\cos\frac{\theta_e}{2}| - \cos\frac{\theta_1}{2}\right) = 0 \quad \Leftrightarrow \quad \begin{cases} \text{or } \sin\frac{\theta_e}{2} = 0 & \Leftrightarrow\ \theta_e = 0 \\ \text{or } \cos\frac{\theta_e}{2} = \pm\cos\frac{\theta_1}{2} & \Leftrightarrow\ \theta_e = \pm\theta_1 \end{cases}$$

   We obtain the equilibrium values $\theta_e = 0, \pm\theta_1$. For the last two values, it must be verified that they do indeed belong to the interval $[-210°, 30°]$.

2. If $\theta_e = -210°$ or $30°$, then $H(\theta_e) = 1$ and equation [10.21] becomes

$$2ka \sin\frac{\theta_e}{2}\left(|\cos\frac{\theta_e}{2}| - \cos\frac{\theta_1}{2}\right) sgn \cos\frac{\theta_e}{2} = N\cos\theta_e$$

   This relationship, where θ_e is known, gives the reaction force N:

$$N = 2ka\frac{\sin\frac{\theta_e}{2}}{\cos\theta_e}\left(|\cos\frac{\theta_e}{2}| - \cos\frac{\theta_1}{2}\right) sgn \cos\frac{\theta_e}{2} \qquad [10.22]$$

   What remains to be verified is that we do indeed have $N \geq 0$. If this is the case, the angle θ_e under consideration is an equilibrium position. If not, it is not an equilibrium position.

In what follows, we assume for the sake of definiteness that the unstretched length of the spring is $\ell_1 = a\sqrt{2}$, that is, that $\theta_1 = \pi/2$. The variation of the potential $\mathcal{V}$ versus the angle θ is then given in the following table, as well as in Figure 10.10.

θ	$\frac{-7\pi}{6}$		$-\pi$		$\frac{-\pi}{2}$		0		$\frac{\pi}{6}$
$\frac{d\mathcal{V}}{d\theta}$	]	+	‖	-	0	+	0	-	[
$\mathcal{V}$	$0.4ka^2$	↗	ka^2	↘	0	↗	$0.17ka^2$	↘	$0.134ka^2$

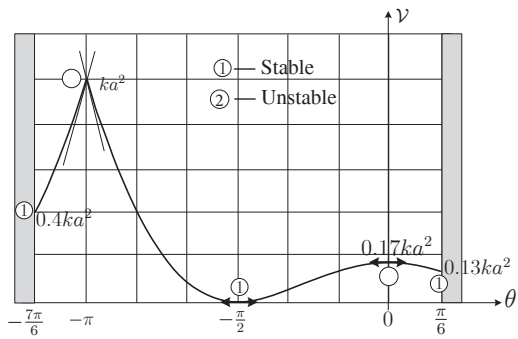


Figure 10.10. *Variation of $\mathcal{V}$ vs. θ*

1. According to case no. 1 above, we have two equilibrium values: $\theta_e = 0$ and $-\pi/2$ (the value $\pi/2$ is excluded).

2. Case no. 2 corresponds to $30°$ or $\theta_e = -210°$. From [10.22], we get the reaction force N:

   – If $\theta_e = 30°$, then $N = \dfrac{4}{\sqrt{3}} ka \sin\dfrac{\pi}{12}\left(\cos\dfrac{\pi}{12} - \cos\dfrac{\pi}{8}\right) = 0.0251ka > 0$: $\theta_e = 30°$ is an equilibrium position. From the physical point of view, as the spring is *stretched* in the position $\theta = 30°$, the force N is positive and the position $30°$ is, indeed, an equilibrium position. If we took the unstretched length of the spring such that $\theta_1 = 15°$, for example, the spring would be *compressed* at $\theta = 30°$, we would have had $N < 0$ and the position $30°$ would *not* be an equilibrium position.

   – If $\theta_e = -210°$, then $N = -\dfrac{4}{\sqrt{3}} ka \sin\dfrac{7\pi}{12}\left(|\cos\dfrac{7\pi}{12}| - \cos\dfrac{\pi}{8}\right) = 1.484ka > 0$: $\theta_e = -210°$ is an equilibrium position.

• The position $\theta = -\pi$ is special. It must be studied separately because the potential $\mathcal{V}$ is not differentiable at point $\theta = -\pi$ and we cannot apply [10.20]. The potential does, however, have the left and right derivatives at this point:

$$\frac{d\mathcal{V}}{d\theta}_{|\theta=-\pi^-} = \sqrt{2}ka^2 \qquad \text{and} \qquad \frac{d\mathcal{V}}{d\theta}_{|\theta=-\pi^+} = -\sqrt{2}ka^2$$

We can show that $\theta = -\pi$ is an unstable equilibrium point.

10.12. Calculating internal efforts

In the previous sections, we considered a mechanical system subjected to a static load and we studied the equilibrium position or positions of the system under this load. As the considered system has some degrees of mobility, it changes its shape under the effect of the load, to arrive at a certain equilibrium position.

In the following sections, we still consider a system of rigid bodies. However, this time the system has no mobility. When an external load is applied, the system does not change shape and if it is in equilibrium, its equilibrium position is the same as before the load was introduced.

Even if the form of the system remains unchanged, efforts are still developed at the internal or external joints (in addition to the efforts internal to each rigid body, which are inaccessible in the framework of rigid bodies mechanics). If the system is isostatic, it is possible to determine the constraint efforts (i) by subdividing the system and studying the equilibrium of each rigid body (this is the Newtonian approach) or (ii) by using the Lagrange equilibrium equations, which is what will be illustrated in the following examples.

10.13. Example: internal efforts in a truss

Consider a Galilean reference frame $R_g = R_0$ endowed with the orthonormal coordinate system $(O; \vec{x}, \vec{y}, \vec{z})$ and a structure that is made up of eight pinned bars subjected to the gravitational field $-g\vec{y}$ (Figure 10.11). The outside bars form a regular hexagon. They are identical and all have the length a and the mass m. The inside bars AA' and BB' are massless. All the joints are assumed to be perfect. One of the vertices of the hexagon is suspended from the point O.

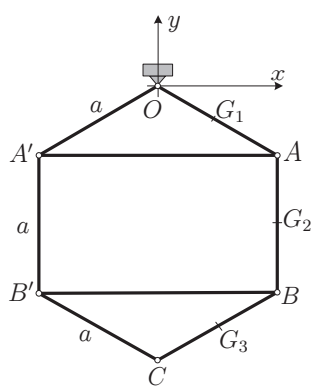


Figure 10.11. *Hexagonal system with eight bars*

The system has no degrees of freedom if it is assumed that the point C remains on the vertical axis $O\vec{y}$. As the bars OA', OA, $B'C$ and BC have a mass, they are simultaneously subjected to tension-compression, bending and shear. Due to their positions, the vertical bars undergo only traction-compression. The horizontal bars $A'A$ and $B'B$, which are massless, also undergo only traction-compression. The objective here is to determine the tension force in these horizontal bars.

10.13.1. *Tension force in bar $A'A$*

To calculate the tension force in bar $A'A$, the idea is to divide this bar into two equal parts $A'D'$ and DA, as shown in Figure 10.12. This operation exhibits the constraint efforts internal to the bar, which are the tension forces denoted by $N_1\vec{x}$ and $-N_1\vec{x}$, applied at D' and D, respectively.

From the initial problem with no mobility, we thus shift to another problem that has one degree of freedom, here taken to be equal to the angle θ positioning the bar OA and defined as in Figure 10.12. We will first consider that N_1 is a given force and we calculate the opening angle θ_e at equilibrium, under the effect of N_1 and of the system's weight. At the end of the calculations, we will determine the force N_1 that is required for θ_e to be equal to $\pi/3$. The found value for N_1 is precisely the tension force desired in the initial problem, where the system at equilibrium has the form of a regular hexagon.

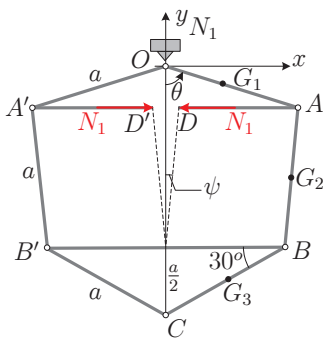


Figure 10.12. *New system with the bar $A'A$ cut*

We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameter: θ.
- No primitive constraint equation.
- Retained parameter: θ.
- No complementary constraint equation.

The equilibrium equation [10.8] can be applied here with $b_i = 0$ and $E^{c(0)} = 0$:

$$\frac{\partial \mathcal{V}}{\partial \theta}_{|\theta_e} = D'_\theta(\theta_e, 0, t), \qquad [10.23]$$

where D'_θ is the generalized force due to forces N_1 at points D, D'.

The position of the system is completely determined by the angle θ. The coordinates of some specific points are given by

Point	x	y
A	$a \sin\theta$	$-a\cos\theta$
B	$a\frac{\sqrt{3}}{2}$	$y_B = y_A - a\cos\psi = -a\cos\theta - a\cos\psi$
C	0	$y_C = y_B - \frac{a}{2} = -a\cos\theta - a\cos\psi - \frac{a}{2}$
D	$a\sin\theta - \frac{a\sqrt{3}}{2}$	$-a\cos\theta$

where the angle ψ denotes the angle of inclination of the vertical bar AB, as shown in Figure 10.12. A simple geometric calculation gives the relation between the angle ψ and θ:

$$\sin\psi = \sin\theta - \frac{\sqrt{3}}{2} \qquad \Leftrightarrow \qquad \psi = \arcsin\left(\sin\theta - \frac{\sqrt{3}}{2}\right)$$

The virtual power of the force at D is

$$\mathcal{P}^*(\text{force at } D) = (-N_1\vec{x}) \cdot \vec{V}^*(D) = -N_1\vec{x} \cdot \frac{\partial \overrightarrow{OD}}{\partial \theta}\dot{\theta}^* = -N_1 a \cos\theta \dot{\theta}^*$$

On account of the symmetry, we get $\mathcal{P}^*(\text{forces at } D \text{ and } D') = -2N_1 a \cos\theta\dot{\theta}^*$. Hence, the generalized force due to the forces at D and at D':

$$D'_\theta = -2N_1 a \cos\theta$$

By denoting the centers of mass of the bars OA, AB, BC by G_1, G_2, G_3, the potential due to weight is

$$\begin{aligned}\mathcal{V} &= 2(mgy_{G_1} + mgy_{G_2} + mgy_{G_3}) + const \\ &= 2(mg\frac{y_A}{2} + mg\frac{y_A + y_B}{2} + mg\frac{y_B + y_C}{2}) + const = 2mg(y_A + y_B + \frac{y_C}{2}) + const \\ &= -mga(5\cos\theta + 3\cos\psi) + const\end{aligned}$$

Finally, the equilibrium equation [10.23] gives

$$mg\left(5\sin\theta_e + 3\sin\psi_e \frac{\partial\psi}{\partial\theta}_{|\theta_e}\right) = -2N_1\cos\theta_e, \qquad [10.24]$$

where ψ_e is the value of ψ when $\theta = \theta_e$. This relationship gives the opening angle θ_e at equilibrium versus N_1. In fact, it turns out that we do not need to know the derivative $\frac{\partial\psi}{\partial\theta} = \frac{\cos\theta}{\sqrt{1 - \left(\sin\theta - \frac{\sqrt{3}}{2}\right)^2}}$ as we will make $\psi_e = 0$ in [10.24].

The desired tension force in the initial problem (without cutting the bar $A'A$) is the force that makes $\theta_e = \frac{\pi}{3}$ (thus, $\psi_e = 0$). Equation [10.24] then gives

$$N_1 = -\frac{5\sqrt{3}}{2}mg$$

The tension force in the bar $A'A$ is negative, which means that the bar is under compression.

REMARK. This is assuming that we wish to solve the problem using Newton's law, that is, through the equilibrium of forces and moments. If, for example, we isolate the node A and study its equilibrium, we must be careful not to replace the bar OA by a force parallel to OA, since the bar is not under pure traction-compression. ■

10.13.2. *Tension force in bar $B'B$*

To calculate the tension force in bar $B'B$, we proceed in a similar manner. We divide this bar into two equal parts $B'E'$ and EB, as shown in Figure 10.13 and we make the tension forces, denoted by $N_2\vec{x}$ and $-N_2\vec{x}$, appear at E' and E, respectively.

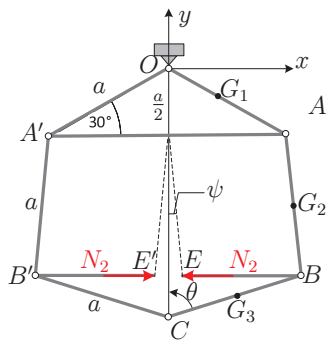


Figure 10.13. *New system with the bar $B'B$ cut*

This time, the degree of freedom introduced, denoted again by θ, is the angle positioning the bar CB. We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameter: θ.
- No primitive constraint equation.
- Retained parameter: θ.
- No complementary constraint equation.

The calculation, similar to those for the bar AA', will not be explained in detail. The coordinates of some specific points are given by

Point	x	y
A	$\frac{a\sqrt{3}}{2}$	$-\frac{a}{2}$
B	$a\frac{\sqrt{3}}{2} + a\sin\psi$	$-\frac{a}{2} - a\cos\psi$
C	0	$y_C = y_B - a\cos\theta = -\frac{a}{2} - a\cos\psi - a\cos\theta$
E	$a\sin\theta - \frac{a\sqrt{3}}{2}$	$y_E = y_B = -\frac{a}{2} - a\cos\psi$

The virtual power of the forces at D and D' is

$$\mathcal{P}^*(\text{forces at } D \text{ and } D') = -2N_2 a\cos\theta\dot{\theta}^*$$

The potential due to the weight is

$$\begin{aligned}\mathcal{V} &= 2(mgy_{G_1} + mgy_{G_2} + mgy_{G_3}) + const = 2mg(y_A + y_B + \frac{y_C}{2}) + const \\ &= -mga(\cos\theta + 3\cos\psi) + const\end{aligned}$$

Thus, the equilibrium equation [10.23] gives

$$N_2 = -\frac{\sqrt{3}}{2}mg\text{: the bar } B'B \text{ is under compression}$$

10.14. Example: internal efforts in a tripod

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}, \vec{y}, \vec{z})$ and a tripod made up of three identical rods, each of which has mass m and length ℓ, pin-jointed at their vertex D and standing on a flat, horizontal ground $O\vec{x}\vec{y}$ (Figure 10.14). The three rods are attached at their midpoints by the ropes $A'B'$, $B'C'$ and $C'A'$ of the same length, such that the support points A, B and C form an equilateral triangle. The rods have the same angle of inclination α with respect to the vertical. A mass M is attached to the vertex D.

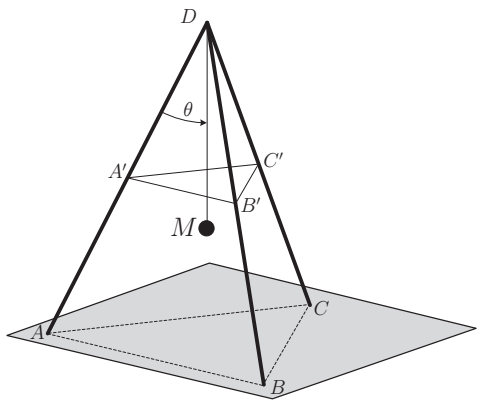


Figure 10.14. *Tripod*

Given the symmetry of the geometry and the load, the tensions in the ropes have the same value, denoted by T. We propose calculating the tension T by following the same steps as in the previous example. In order to do this, we do away with the three ropes, replacing them by the tensions T exerted at the points A', B', C', and we thus create a new system, where the three rods may move (Figure 10.15). By assuming that the geometric symmetry is conserved, that is, each rod moves while remaining in its original plane (the plane defined by the vertical axis $O\vec{z}$ and the initial position of the rod), the position of the system is defined by a single parameter, chosen to be equal to the inclination angle θ of the rods with respect to the vertical. We will first consider that T is a given force and we calculate the angle of inclination θ_e at equilibrium, under the effect of T and of the system's weight. The tension T, which we wished to find in the initial problem, is ultimately found by forcing θ_e to be equal to the given angle α.

The equilibrium equation is still [10.23], where D'_θ is, here, the generalized force due to the tensions N in the ropes. The position of the system is completely determined by θ. In particular, the coordinates of specific points are given by

Point	x	z
A	$\ell \sin\theta$	0
D	0	$\ell \cos\theta$
A'	$\ell/2 \sin\theta$	$\ell/2 \cos\theta$

Owing to symmetry, the virtual power of the tensions is equal to three times that of the tensions applied at point A' (see Figure 10.15). We thus only have to calculate the virtual power of the

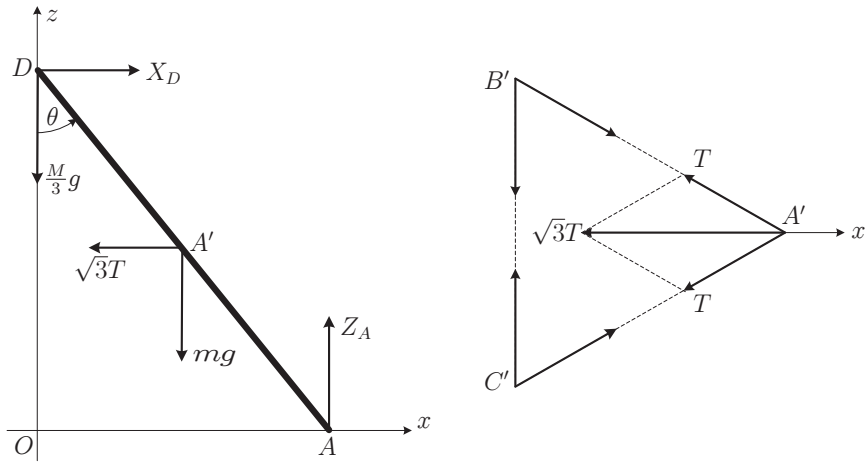


Figure 10.15. *New system with the degree of freedom θ*

resultant force $\sqrt{3}T$ of the two tensions at A':

$$\mathcal{P}^*(\text{tensions at } A') = (-\sqrt{3}T\vec{x}) \cdot \vec{V}^*(A') = -\sqrt{3}T\vec{x} \cdot \frac{\partial \overrightarrow{OA'}}{\partial \theta}\dot{\theta}^* = -\frac{\sqrt{3}}{2}T\ell\cos\theta\dot{\theta}^*$$

Hence, the generalized force due to the tensions at A', B' and C':

$$D'_\theta = -\frac{3\sqrt{3}}{2}T\ell\cos\theta$$

The potential of the system is due to its weight:

$$\mathcal{V} = Mgz_D + 3mgz'_A + const = Mg\ell\cos\theta + \frac{3}{2}mg\ell\cos\theta + const$$

Finally, the equilibrium equation [10.23] gives T as a function of the angle θ_e at equilibrium:

$$T = \frac{2M + 3m}{3\sqrt{3}}g\tan\theta_e$$

The desired tension in the initial problem (without cutting the ropes $A'B'$, $B'C'$ and $C'A'$) is that force which makes $\theta_e = \alpha$:

$$T = \frac{2M + 3m}{3\sqrt{3}}g\tan\alpha$$

11

Revision Problems

In this chapter, we treat some additional problems as review problems in Lagrangian mechanics. The first part of this chapter will cover equilibrium problems and the second part will study dynamic problems. For the dynamic problems, we will first consider plane motions and then motions in space.

11.1. Equilibrium of two rods

Consider a planar system $\mathcal{S}$ made up of two rods AB and BC, at rest in a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ with $\vec{x}_0$ downward vertical (Figure 11.1):

– AB: a homogeneous rod of $2a$, mass $2m$ and center of mass D;

– BC: a homogeneous rod of length $2a$, mass m and center of mass E;

– the pin joints at the ends of the rods are perfect. The end A is constantly located at O, while the end C slides without friction on the axis $O\vec{y}_0$.

The system is in equilibrium under the action of the gravity field $g\vec{x}_0$ and a force $F\vec{y}_0$ at C applied on the rod BC.

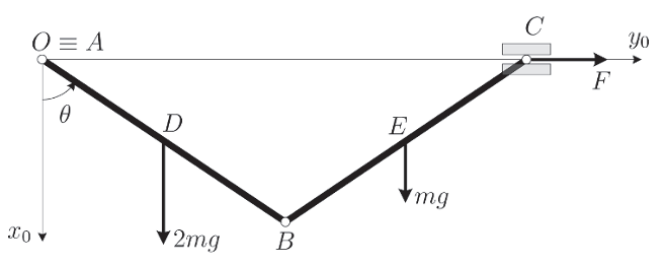


Figure 11.1. *Equilibrium of two rods*

We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameter: $\theta \equiv (\vec{x}_0, \overrightarrow{AB})$.
- No primitive constraint equation.

Lagrangian Mechanics: An Advanced Analytical Approach,
First Edition. Anh Le van and Rabah Bouzidi.

- Retained parameter : θ.
- No complementary constraint equation.

The joints are perfect, the parameterization is independent and the given forces are derivable from a potential. The equilibrium equation is, thus, reduced to the stationary condition [10.10] of the potential: $\frac{\partial \mathcal{V}}{\partial \theta}_{|\theta_e} = 0$.

Let us calculate the potential of the given forces applied on the system:

$$\begin{aligned} \mathcal{V} &= -m\vec{g}\cdot\overrightarrow{OC} - 2m\vec{g}\cdot\overrightarrow{OD} - \vec{F}\cdot\overrightarrow{OB} \\ &= -3mga\cos\theta - 4Fa\sin\theta \end{aligned}$$

The equilibrium equation thus gives

$$3mga\sin\theta_e - 4Fa\cos\theta_e = 0 \quad \Rightarrow \quad \tan\theta_e = \frac{4F}{3mg}$$

REMARK. If we wished to calculate the reactions $\vec{R}_O$ on O and $X_B\vec{x}_0$ on B due to the supports, we should have adopted the more complicated parameterization given below, where the constraint equations associated with the reaction forces in question are classified as complementary equations:

PARAMETERIZATION.

- Primitive parameters:
  - we define the position of rod AB in the plane by the coordinatesx_A, y_A of end A and the angle $\theta = (\widehat{\overrightarrow{Ox}, \overrightarrow{AB}})$,
  - once the position of AB is defined, we decide to define the position of rod BC by the coordinate x_C of the end C.

  We have, therefore, four primitive parameters: x_A, y_A, θ, x_C.
- No primitive constraint equation.
- Retained parameters: x_A, y_A, θ, x_C.
- Complementary constraint equations: $x_A = y_A = 0$, and $x_C = 0$. ■

11.1.1. *Analysis using Newton's law*

- Calculation of the reaction X_B: The moment equilibrium of system $\mathcal{S}$ about axis $O\vec{z}_0$ is given as:

$$-2mga\sin\theta - mg3a\sin\theta - X_B 4a\sin\theta = 0 \qquad \Rightarrow \qquad X_B = -\frac{5}{4}mg$$

- Moment equilibrium condition for AB about axis $A\vec{z}_0$ is given as:

$$-mga\sin\theta - X_B 2a\sin\theta - F2a\cos\theta = 0$$

  By taking into account the value of X_B, we have

$$\tan\theta = \frac{4F}{3mg}$$

The action of the support O on OA is obtained by writing that the resultant force on $\mathcal{S}$ is zero:

$$\sum \vec{F}_{ext\to\mathcal{S}} = \vec{0} \quad \Rightarrow \quad \vec{R}_O + 3mg\vec{x}_0 + F\vec{y}_0 + X_B\vec{x}_0 = 0$$

Hence, $\vec{R}_O = -7/4mg\vec{x}_0 - F\vec{y}_0$.

11.2. Equilibrium of an elastic chair

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}, \vec{y}, \vec{z})$ and a mechanical system composed of two identical, rigid rods AC and BD with length l and pinned at point E. The two rods are connected at their ends B and C through a spring of stiffness k. The rod AB is pinned at A to the support and the end D slides horizontally parallel to $\vec{x}$. All the joints are perfect.

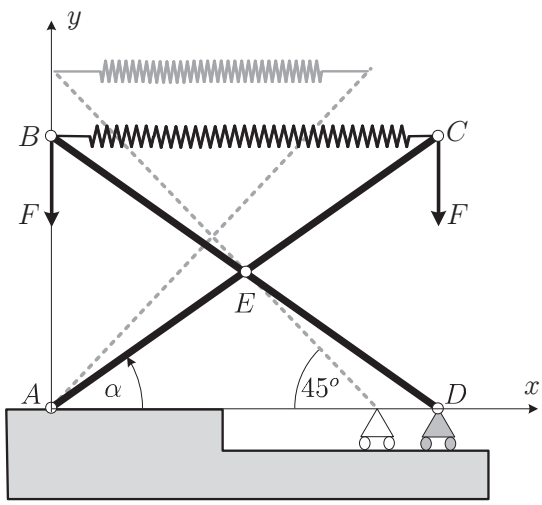


Figure 11.2. *Equilibrium of an elastic chair*

The position of the system is defined by the angle $\alpha \equiv (\vec{x}, \overrightarrow{AB})$. It is assumed that the spring is unstretched when $\alpha = 45°$. When we apply two identical forces $\vec{F} = -F\vec{y}$ at points B and C, the system moves down to an equilibrium position that we wish to calculate. We use the following parameterization:

PARAMETERIZATION.

- Primitive parameters: α.
- No primitive constraint equation.
- Retained parameter: α.
- No complementary constraint equation.

The joints are perfect, the parameterization is independent and the given forces are derivable from a potential. The equilibrium equation is, thus, given by [10.10]: $\dfrac{\partial \mathcal{V}}{\partial \alpha}_{|\alpha_e} = 0$.

Let us calculate the potential of the given forces applied to the system:

– As the elongation of the spring is given by

$$\Delta l = l_{current} - l_{initial} = l\cos\alpha - l\cos 45^o = l\left(\cos\alpha - \frac{\sqrt{2}}{2}\right),$$

the potential of the spring is written as

$$\mathcal{V}_{spring} = \frac{1}{2}k\Delta l^2 + cte = \frac{1}{2}kl^2\left(\cos\alpha - \frac{\sqrt{2}}{2}\right)^2 + const$$

– The potential of the given forces applied at B and C is

$$\mathcal{V}_F = -\vec{F}\cdot\overrightarrow{AB} - \vec{F}\cdot\overrightarrow{AC} + const = 2Fl\sin\alpha + const$$

The total potential is, thus

$$\mathcal{V} = \mathcal{V}_{spring} + \mathcal{V}_F = \frac{1}{2}kl^2\left(\cos\alpha - \frac{\sqrt{2}}{2}\right)^2 + 2Fl\sin\alpha + const$$

The equilibrium equation thus gives

$$\left.\frac{\partial\mathcal{V}}{\partial\alpha}\right|_{\alpha=\alpha_e} = -kl^2\left(\cos\alpha - \frac{\sqrt{2}}{2}\right)\sin\alpha + 2Fl\cos\alpha = 0 \quad\Rightarrow\quad \frac{F}{kl} = \frac{1}{2}\left(\cos\alpha - \frac{\sqrt{2}}{2}\right)\tan\alpha$$

We may also determine the stability of the equilibrium with the help of the second derivative of the potential:

$$\begin{aligned}\frac{\partial^2\mathcal{V}}{\partial\alpha^2} &= kl^2\left(\sin^2 - \cos^2\alpha\right) + \frac{\sqrt{2}}{2}kl^2\cos\alpha - 2Fl\sin\alpha \\ &= kl^2\left[\left(\sin^2 - \cos^2\alpha\right) + \frac{\sqrt{2}}{2}\cos\alpha - \left(\cos\alpha - \frac{\sqrt{2}}{2}\right)\tan\alpha\sin\alpha\right]\end{aligned}$$

Figure 11.3 shows the dimensionless force F/kl as well as the second derivative of the potential versus the angle α. The equilibrium is stable in the interval $[27°, 45°]$, where the second derivative of potential is positive. It is unstable in the interval $[0°, 27°]$.

11.3. Equilibrium of a dump truck

Here, we study the equilibrium of a dump truck in a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}, \vec{y}, \vec{z})$ with $\vec{y}$ defining the upward vertical. The truck comprises several rigid bodies:

– the rigid body S_1 made up of the chassis and the cabin,

– the rigid body S_2 made up of the tipper and the material being transported, with mass m and center of mass G_2, such that $\overrightarrow{OG_2} = \frac{h}{2}\left[(3\cos\theta - \sin\theta)\,\vec{x} + (3\sin\theta + \cos\theta)\,\vec{y}\right]$, where $\theta \equiv (\vec{x}, \overrightarrow{OD})$,

– a lifting system, made up of two arms CD and CG_1, connecting the two above-mentioned rigid bodies. These arms are assumed to be weightless and equipped with an actuator AB (this deformable actuator is *not* a component of the system being studied). It is assumed that $CB = CA = h$, $CD = CG_1 = 2h$.

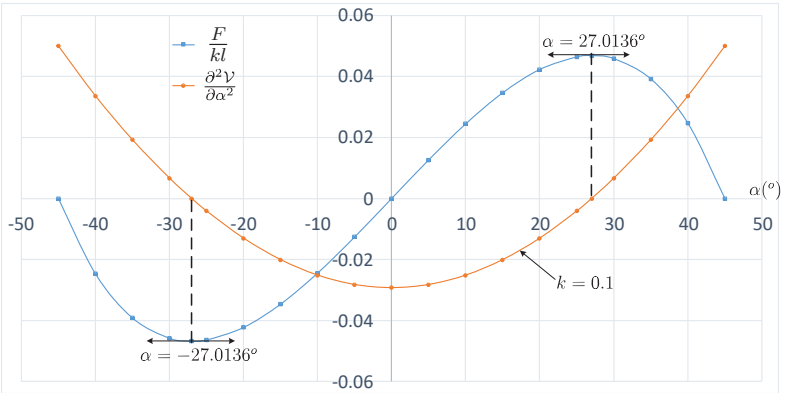


Figure 11.3. *Force and second derivative of the potential versus angle α.*

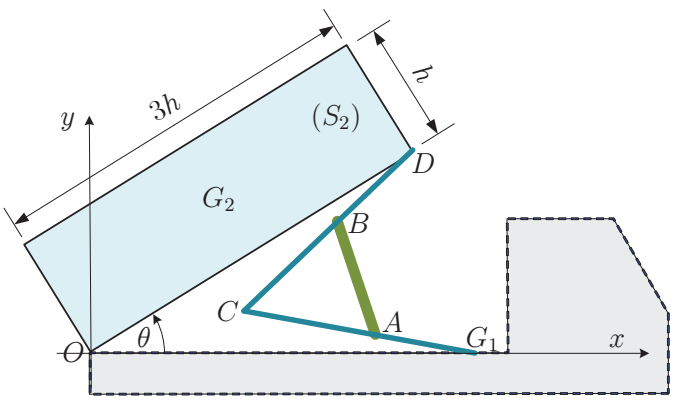


Figure 11.4. *Equilibrium of a dump truck*

The whole system is subjected to the gravity field $-g\vec{y}$. We wish to determine the force in the actuator AB as a function of the equilibrium position θ of the tipper.

PARAMETERIZATION.

- Primitive parameter: $\theta \equiv (\vec{x}, \overrightarrow{OD})$.
- No primitive constraint equation.
- Retained parameter: θ.
- No complementary constraint equation.

The virtual power of the forces $\vec{F}_A$ and $\vec{F}_B$ in the actuator, exerted at points A and B, is

$$\mathscr{P}^* \text{ (actuator)} = \vec{F}_A \cdot \vec{V}^*(A) + \vec{F}_B \cdot \vec{V}^*(B)$$

Further, the equilibrium of the actuator implies that $\vec{F}_A = -\vec{F}_B$ and $\vec{F}_A$ is parallel to $\overrightarrow{AB}$. By writing $\vec{F}_B = F\vec{i}$, where $\vec{i}$ denotes the unit vector orienting $\overrightarrow{AB}$, we have

$$\begin{aligned}
\mathscr{P}^* \text{ (actuator)} &= \vec{F}.\left[\vec{V}^*(B) - \vec{V}^*(A)\right] = \vec{F}.\frac{d\overrightarrow{AB}}{d\theta}\dot{\theta}^* \\
&= F\vec{i}.\frac{d(AB\vec{i})}{d\theta}\dot{\theta}^* && \text{where } \frac{d(AB\vec{i})}{d\theta} = AB\frac{d\vec{i}}{d\theta} + \frac{dAB}{d\theta}\vec{i} \\
&= F.\frac{dAB}{d\theta}\dot{\theta}^* && \text{taking into account } \vec{i}.\frac{d\vec{i}}{d\theta} = 0 \\
&= \frac{d}{d\theta}\left(3h\sin\frac{\theta}{2}\right)\dot{\theta}^* = F\left(\frac{3}{2}h\cos\frac{\theta}{2}\right)\dot{\theta}^* && \text{knowing that } AB = \frac{1}{2}G_1D = 3h\sin\frac{\theta}{2}
\end{aligned}$$

To obtain the virtual power of the weight of tipper S_2, applied at G_2, let us calculate the virtual velocity of the center of mass G_2:

$$\vec{V}^*_{02}(G_2) = \frac{\partial\overrightarrow{OG_2}}{\partial\theta}\dot{\theta}^* = \frac{h}{2}\left[(-3\sin\theta - \cos\theta)\vec{x} + (3\cos\theta - \sin\theta)\vec{y}\right]\dot{\theta}^*$$

Hence, the virtual power of the weight of the tipper is given as

$$\mathscr{P}^* \text{ (weight)} = m\vec{g}\cdot\vec{V}^*_{02}(G_2) = -mg\frac{h}{2}\left(3\cos\theta - \sin\theta\right)\dot{\theta}^*$$

The total virtual power of the external forces is given as

$$\begin{aligned}
\mathscr{P}^* \text{ (total)} &= \mathscr{P}^* \text{ (actuator)} + \mathscr{P}^* \text{ (weight)} \\
&= \left[F\left(\frac{3}{2}h\cos\frac{\theta}{2}\right) - mg\frac{h}{2}\left(3\cos\theta - \sin\theta\right)\right]\dot{\theta}^* = Q_\theta\dot{\theta}^*
\end{aligned}$$

The Lagrange equations are reduced to

$$Q_\theta = 0 \qquad \Rightarrow \qquad F = \frac{mg}{3}\frac{3\cos\theta - \sin\theta}{\cos\theta/2}$$

Figure 11.5 represents the ratio of the actuator force to the tipper's weight versus angle θ. The angle at which the center of mass G_2 of the tipper passes above the pivot O is $\theta_1 = \pi/2 - \tan^{-1}(1/3) = 71.57°$. The actuator force is then reversed.

11.4. Equilibrium of a set square

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and a rigid body S made up of two rods $(B_1), (B_2)$ welded to a right angle between them. The rod (B_1) has length l and mass m, whereas the rod (B_2) has length $2l$ and mass $2m$. A perfect pivot joint parallel to the axis $O\vec{z}_0$ allows the rigid body to rotate about this axis. The rigid body is at equilibrium in the downward gravity field $g\vec{x}_0$.

We define the orthogonal unit vectors $\vec{x}_1, \vec{y}_1$ as shown in the figure, and we use the following parameterization:

PARAMETERIZATION.

- Primitive parameter: $\theta \equiv (\widehat{\vec{x}_0, \vec{x}_1})$.
- No primitive constraint equation.
- Retained parameter: θ.
- Complementary constraint equation.

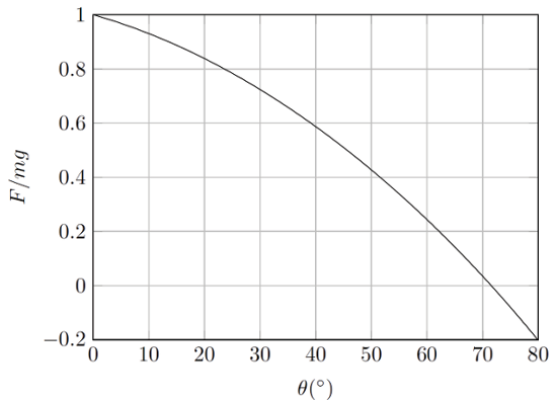


Figure 11.5. *Ratio F/mg versus θ*

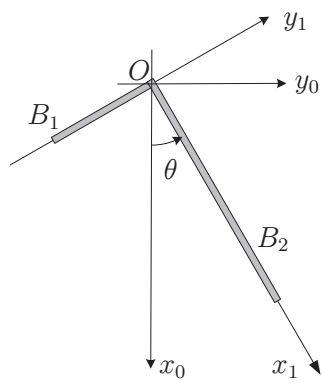


Figure 11.6. *Equilibrium of a set square*

The position vector of the center of mass G is

$$\overrightarrow{OG} = \frac{m_1\overrightarrow{OG_1} + m_2\overrightarrow{OG_2}}{m_1 + m_2} = \frac{2l}{3}\vec{x}_1 - \frac{l}{6}\vec{y}_1$$

Here, only the weight does work. We thus have

$$\begin{aligned}\mathscr{P}^*\,(\text{weight } \to S) &= 3m\vec{g}\cdot\vec{V}^*_{R_0}(G) = 3mg\vec{x}_0\cdot\left(\frac{l}{6}\vec{x}_1 + \frac{2l}{3}\vec{y}_1\right)\dot{\theta}^* \\ &= 3mgl\left(\frac{1}{6}\cos\theta - \frac{2}{3}\sin\theta\right)\dot{\theta}^* = Q_\theta\dot{\theta}^*\end{aligned}$$

At equilibrium, Lagrange's equations reduce to

$$Q_\theta = 0 \qquad \Rightarrow \qquad \frac{1}{6}\cos\theta - \frac{2}{3}\sin\theta = 0 \qquad \Rightarrow \qquad \theta = \tan^{-1}\left(\frac{1/6}{2/3}\right) = 14.03^\circ$$

This equilibrium position can also be obtained easily using statics laws, knowing that at equilibrium, the center of mass G of S must be located above the pivot O.

11.5. Motion of a metronome

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ with $\vec{y}_0$ upward vertical and a metronome modeled as a two-dimensional system $\mathcal{S}$ moving in the plane $O\vec{x}_0\vec{y}_0$. The system is made up of three rigid bodies:

– a base S, schematized as a homogeneous rectangle, with center O and mass m;

– a rod OA, of length $4a$, of negligible mass, connected at O to S through a perfect pivot;

– a homogeneous disk D, with center C, radius a and mass m, welded to OA such that $OC = 2a$.

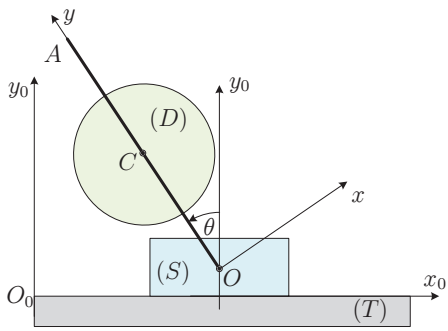


Figure 11.7. *Motion of a metronome*

We define $\vec{y}$ as the unit vector orienting $\overrightarrow{OA}$, $\vec{x} = \vec{z}_0 \times \vec{y}$ and the angle $\theta \equiv (\widehat{\vec{y}_0, \vec{y}})$. A mechanism, not represented in the figure, allows S to exert on OA a restoring torque $\vec{\Gamma} = -3mga \sin\theta \vec{z}_0$. The base S is laid on a table T, whose upper surface is the plane $O\vec{x}_0\vec{y}_0$. The system is subjected to the gravity field $-g\vec{y}_0$.

We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: θ.
- No primitive constraint equation.
- Retained parameter: θ.
- No complementary constraint equation.

11.5.1. *Equation of motion*

We first assume that the base S is fixed on the table T. With the velocity of the center C being $\vec{V}_{0D}(C) = -2a\dot{\theta}\vec{x}$, the kinetic energy of the system is written as

$$E^c_{0D} = \frac{9}{4} m a^2 \dot{\theta}^2$$

The potential $\mathcal{V}_1$ due to the weight of D is $\mathcal{V}_1 = 2mga\cos\theta + const$. The virtual power of the torque $\vec{\Gamma}$ being $\mathscr{P}^*\left(\vec{\Gamma}\right) = -3mga\sin\theta\dot{\theta}^*$, the potential $\mathcal{V}_2$ of the torque is $\mathcal{V}_2 = -3mga\cos\theta + const$. Hence, the total potential of the system:

$$\mathcal{V} = -mga\cos\theta + const \qquad [11.1]$$

Lagrange's equation corresponding to parameter θ is written as (conservative system)

$$\frac{9}{2}ma^2\ddot{\theta} + mga\sin\theta = 0 \qquad [11.2]$$

11.5.2. *First integral*

All the requirements are met for the energy integral [9.11], $E^c + \mathcal{V} = const$:

$$\frac{9}{4}ma^2\dot{\theta}^2 - mga\cos\theta = const$$

We could also have obtained this by multiplying [11.2] by $\dot{\theta}$ and then integrating the result with respect to time. The motion is an oscillation of large amplitude.

Let us assume the initial conditions $\theta_0 = \frac{\pi}{3}$ and $\dot{\theta}_0 = 0$. The above first integral then becomes

$$\dot{\theta}^2 = \frac{4g}{9a}\left(\cos\theta - \frac{1}{2}\right)$$

The amplitude θ_{max} of the oscillations is calculated using the condition $\dot{\theta}_{|\theta_{max}} = 0$ (reversal point of the metronome). We find $\theta_{max} = \frac{\pi}{3}$.

11.5.3. *The case of small oscillations*

If we now consider the case of small oscillations, the motion equation [11.2] is reduced to

$$\ddot{\theta} + \frac{9g}{2a}\theta = 0$$

This is a periodic oscillatory motion with the circular frequency $\omega = \sqrt{\frac{9g}{2a}}$.

11.5.4. *Case where the base S may slide without friction on the table T*

From now on, it is assumed that the base S may slide without friction on the table T.

PARAMETERIZATION.

- Primitive parameters: θ and $\lambda \equiv \overrightarrow{O_0O} \cdot \vec{x}_0$.
- No primitive constraint equations.
- Retained parameters: θ, λ.
- No complementary constraint equation.

The kinetic energy is

$$E^c_{0S} = E^c_{0S} + E^c_{0D} = m\dot{\lambda}^2 + \frac{9}{4}ma^2\dot{\theta}^2 - 2ma\dot{\lambda}\dot{\theta}\cos\theta$$

The potential has the same expression [11.1] as in the first parameterization. Hence, Lagrange's equations are

$$\mathscr{L}_\lambda : \quad \frac{d}{dt}(\dot{\lambda} - a\dot{\theta}\cos\theta) = 0 \quad \Leftrightarrow \quad \dot{\lambda} = a\dot{\theta}\cos\theta + const$$

$$\mathscr{L}_\theta : \quad \frac{9}{4}a\ddot{\theta} - \frac{d}{dt}(\dot{\lambda}\cos\theta) - \dot{\lambda}\dot{\theta}\sin\theta + \frac{g}{2a}\sin\theta = 0$$

Using $\mathscr{L}_\lambda$ to eliminate $\dot{\lambda}$ in $\mathscr{L}_\theta$, in favor of $\dot{\theta}$, we arrive at the equation of motion governing θ:

$$\frac{9}{4}\ddot{\theta} + \underbrace{\dot{\theta}^2 \sin\theta\cos\theta - \ddot{\theta}\cos^2\theta}_{-\frac{d}{dt}\dot{\theta}^2\cos^2\theta} + \frac{g}{2a}\sin\theta = 0 \qquad [11.3]$$

11.5.5. *First integral*

We once again have the energy integral [9.11], $E^c + \mathcal{V} = const$, which is written as

$$\dot{\lambda}^2 + \frac{9}{4}a^2\dot{\theta}^2 - 2a\dot{\lambda}\dot{\theta}\cos\theta - ga\cos\theta = const$$

Using $\mathscr{L}_\lambda$ to eliminate $\dot{\lambda}$, we arrive at the so-called equation for the principal parameter θ:

$$\dot{\theta}^2\left(\frac{9}{4} - \cos^2\theta\right) = \frac{g}{a}\cos\theta + const$$

This same result may also be obtained by multiplying [11.3] by $\dot{\theta}$ and then integrating the result with respect to time.

11.6. Analysis of a hemispherical envelope

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ with $\vec{x}_0$ downward vertical and a rigid body S, which has the form of a homogeneous hemispherical envelope, with center O, radius a, mass m and center of mass G. The rigid body S is in no-slipping contact at a point I with the horizontal plane $O\vec{y}_0\vec{z}_0$.

We define the orthonormal coordinate system $(O; \vec{x}, \vec{y}, \vec{z})$ attached to S such that $O\vec{x}$ is the axis of revolution of S and we assume a planar motion of S, such that the plane $O\vec{x}\vec{y}$ remains identical to the plane $O\vec{x}_0\vec{y}_0$.

The rigid body is subjected to the gravity field $g\vec{x}_0$.

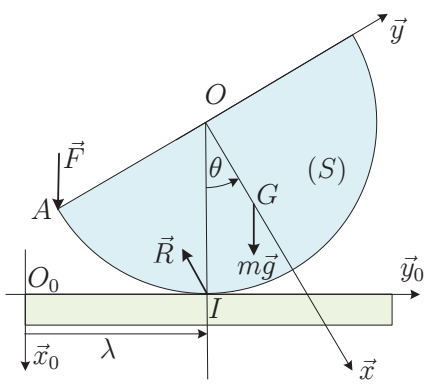


Figure 11.8. *Equilibrium and motion of a hemispherical envelope*

Let $\eta, \lambda = \overline{O_0 I}$ denote the coordinates of the point O with respect to the coordinate system $(O_0; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ and θ the angle $(\widehat{\vec{x}_0, \vec{x}}) = (\widehat{\vec{y}_0, \vec{y}})$. The contact at I between the envelope and the

horizontal plane is expressed by $\eta = a$. The relative velocity at the contact point I between the rigid body S and the plane $O\vec{y}_0\vec{z}_0$ is

$$\vec{V}_{0S}(I) = \vec{V}_{0S}(I) + \vec{\Omega}_{0S} \times \overrightarrow{OI} = \dot{\eta}\vec{x}_0 + (\dot{\lambda} + a\dot{\theta})\vec{y}_0 \underset{\eta=a}{=} (\dot{\lambda} + a\dot{\theta})\vec{y}_0$$

The no-slip condition at I is expressed by $\dot{\lambda} + a\dot{\theta} = 0$. This is a semi-holonomic relationship that can be integrated into

$$\lambda + a\theta = \lambda_0 + a\theta_0,$$

where λ_0, θ_0 are the known initial values of λ, θ. We will take this relationship as a primitive equation in order to eliminate λ in favor of θ. Thus, the parameterization used is as follows:

PARAMETERIZATION.

- Primitive parameters: η, λ, θ.
- Primitive constraint equation: $\eta = a$, $\lambda = \lambda_0 + a(\theta_0 - \theta)$.
- Retained parameter: θ.
- No complementary constraint equation.

11.6.1. *Studying the static equilibrium*

A vertical force $\vec{F} = F\vec{x}_0$ is applied at point A on the edge of S, defined by $\overrightarrow{OA} = -a\vec{y}$. We wish to find the equilibrium position of S.

As the contact point I does not slip, the contact joint is perfect. Let us calculate the virtual power of the given forces $\vec{F}$ and $m\vec{g}$. The virtual velocities at the points of application A, G are (recall that $OG = a/2$ for a hemispherical envelope) given as:

$$\begin{aligned}
\overrightarrow{O_0A} &= \lambda\vec{y}_0 - a\vec{x}_0 - a\vec{y} &&= [\lambda_0 + a(\theta_0 - \theta)]\,\vec{y}_0 - a\vec{x}_0 - a\vec{y} &&\Rightarrow \vec{V}^*_{0S}(A) = -a\dot{\theta}^*\vec{y}_0 + a\dot{\theta}^*\vec{x} \\
\overrightarrow{O_0G} &= \lambda\vec{y}_0 + \tfrac{a}{2}\vec{x} &&= [\lambda_0 + a(\theta_0 - \theta)]\,\vec{y}_0 + \tfrac{a}{2}\vec{x} &&\Rightarrow \vec{V}^*_{0S}(G) = -a\dot{\theta}^*\vec{y}_0 + \tfrac{a}{2}\dot{\theta}^*\vec{y}
\end{aligned}$$

Hence, the virtual power of the given forces is written as:

$$\mathscr{P}^*(S) = m\vec{g} \cdot \vec{V}^*_{0S}(G) + \vec{F} \cdot \vec{V}^*_{0S}(A) = \left[-\frac{mga}{2}\sin\theta + Fa\cos\theta\right]\dot{\theta}^* = Q_\theta\dot{\theta}^* \qquad [11.4]$$

The equilibrium equation $Q_\theta = 0$ gives the equilibrium angle:

$$\tan\theta = \frac{2F}{mg}$$

As expected, the result is independent of the initial value λ_0 (as concerns the initial value θ_0, it is, of course, equal to the above-found equilibrium value). We can also determine the angle of equilibrium θ_0 by using the potential of the two given forces $\vec{F}$ and $m\vec{g}$:

$$\begin{aligned}
\mathcal{V} &= -m\vec{g} \cdot \overrightarrow{OG} - \vec{F} \cdot \overrightarrow{OA} + const \\
&= -\frac{mga}{2}\cos\theta - Fa\sin\theta + const
\end{aligned}$$

The potential is stationary at equilibrium:

$$\frac{\partial\mathcal{V}}{\partial\theta} = \frac{mga}{2}\sin\theta - Fa\cos\theta = 0 \quad \Rightarrow \quad \tan\theta = \frac{2F}{mg}$$

We, thus, clearly arrive at the same result as above.

11.6.2. *Studying the oscillatory motion*

As the rigid body S is initially at rest at $\theta_0 = \pi/3$, we suddenly remove the force $\vec{F}$ at the instant $t = 0$. Under the effect of gravity alone, we then have large oscillations between θ_0 and $-\theta_0$.

Knowing that the moment of inertia of S with respect to axis $G\vec{z}$ is $I_S(G\vec{z}) = 5/12ma^2$, we can calculate the kinetic energy of S:

$$E_{0S}^c = \frac{1}{6}\left(5 - 3\cos\theta\right) ma^2\dot{\theta}^2$$

The generalized force Q_θ taken from [11.4] is still valid, with $F = 0$ here:

$$Q_\theta = -\frac{mga}{2}\sin\theta$$

Lagrange's equation gives the differential equation of motion governing θ:

$$\frac{1}{3}\left(5 - 3\cos\theta\right) ma^2\ddot{\theta} + \frac{3}{2}ma^2\dot{\theta}^2\sin\theta = -\frac{mga}{2}\sin\theta$$

As expected, the motion in θ is independent of the initial value λ_0.

If we consider small oscillations of S around $\theta = 0$, we have the following approximations:

$$\sin\theta \approx \theta, \qquad \cos\theta \approx 1, \qquad \dot{\theta}^2 \approx 0$$

The equation of motion is transformed into a second-order linear differential equation:

$$\ddot{\theta} + \frac{3g}{4a}\theta = 0$$

The motion is oscillatory, with the circular frequency $\omega = \sqrt{\dfrac{3g}{4a}}$.

11.7. A block rolling on a cylinder

We study a parallelepiped (P) moving in a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$ with $\vec{y}_0$ upward vertical. The parallelepiped is homogenous, with mass m and center of mass G, with length l and height h. It is in non-slipping contact at a point I with a cylinder (C) of radius R, fixed in R_g. The whole system is subjected to gravity $-g\vec{y}_0$.

The *a priori* position of the parallelepiped (P) in the plane $O\vec{x}_0\vec{y}_0$ is defined by three parameters. To be more explicit, let us introduce the following notations (Figure 11.10):

- H is the projection of O on the line (AB) (the lower face of (P)),
- d is the distance OH,
- $\vec{x}, \vec{y}$ are the unit vectors orienting $\overrightarrow{AB}$ and $\overrightarrow{OH}$, respectively.
- θ is the angle $(\widehat{\vec{y}_0, \vec{y}})$ and λ is the abscissa of the center of mass G with respect to the coordinate system $(H; \vec{x}, \vec{y})$.

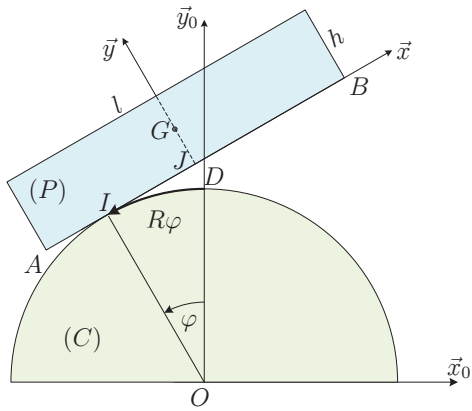


Figure 11.9. *A block rolling on a cylinder*

To define the position of (P):

– we need φ and d to define the position of the line (AB),

– once we know the position of (AB), we use λ to define the position of the center G along (AB).

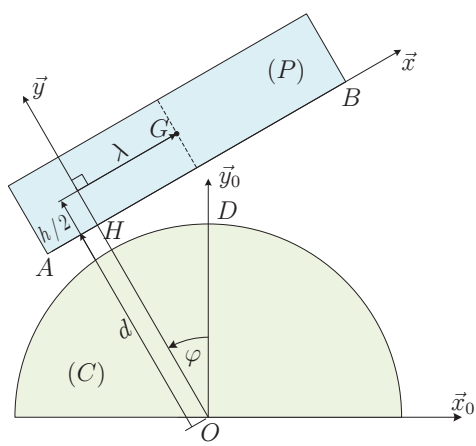


Figure 11.10. *The primitive parameters of the block*

Let us calculate the velocity of the mass center G of (P) with respect to R_0:

$$\overrightarrow{OG} = \overrightarrow{OI}+\overrightarrow{IJ}+\overrightarrow{JG} = \lambda\vec{x}+(R+\frac{h}{2})\vec{y} \qquad \Rightarrow \qquad \vec{V}_{0P}(G) = \dot{\lambda}\vec{x}-(R+\frac{h}{2})\dot{\varphi}\vec{x}+\lambda\dot{\varphi}\vec{y} \quad [11.5]$$

From this, we derive the slip velocity at the contact point I between the parallelepiped (P) and the cylinder (C):

$$\begin{aligned}\vec{V}_{0P}(I) &= \vec{V}_{0P}(G) + \vec{\Omega}_{0P} \times \overrightarrow{GI} \\ &= \dot{\lambda}\vec{x} - (R + \frac{h}{2})\dot{\varphi}\vec{x} + \lambda\dot{\varphi}\vec{y} + \dot{\varphi}\vec{z}_0 \times (-\lambda\vec{x} - \frac{h}{2}\vec{y}) \\ &= (\dot{\lambda} - R\dot{\varphi})\vec{x}\end{aligned}$$

The no-slip condition at I is, thus, expressed by $\dot{\lambda} = R\dot{\varphi}$, a semi-holonomic relationship that, after time integration, gives

$$\lambda = R(\varphi - \varphi_0) + \lambda_0,$$

where λ_0, φ_0 are the known initial values of λ, φ. Let us assume that at the initial instant, (P) is in the horizontal position, the contact point I is at the midpoint J of the side (AB) and also the vertex S of (C). Thus, $\lambda_0 = 0$ and $\varphi_0 = 0$ and, therefore,

$$\lambda = R\varphi$$

This relationship will be taken as a primitive equation in order to eliminate λ in favor of φ. Thus, the parameterization used is as follows:

PARAMETERIZATION.

- Primitive parameters: λ, d, φ.
- Primitive constraint equations: $d = R$, $\lambda = R\varphi$.
- Retained parameter: φ.
- No complementary constraint equation.

Using this parameterization, relationship [11.5] becomes

$$\overrightarrow{OG} = R\varphi\vec{x} + (R + \frac{h}{2})\vec{y} \qquad \text{and} \qquad \vec{V}_{0P}(G) = \left(R\varphi\vec{y} - \frac{h}{2}\vec{x}\right)\dot{\varphi} \qquad [11.6]$$

11.7.1. *Studying the equilibrium*

Let us first study the equilibrium positions of (P) and their stability.

As there is no slipping contact at I, the contact joint is perfect. Let us calculate the virtual power of the weight of (P). The virtual velocity of the center G can be easily obtained from [11.6]:

$$\vec{V}^*_{0P}(G) = \left(R\varphi\vec{y} - \frac{h}{2}\vec{x}\right)\dot{\varphi}^*$$

The virtual power of the weight of (P) is, therefore

$$\mathscr{P}^*(P) = m\vec{g} \cdot \vec{V}^*_{0P}(G) = mg\left(\frac{h}{2}\sin\varphi - R\varphi\cos\varphi\right)\dot{\varphi}^* = Q_\varphi\dot{\varphi}^*$$

The equilibrium equation $Q_\varphi = 0$ gives the equilibrium angle:

$$\tan\varphi = \frac{2R}{h}\varphi$$

whose solution in the interval $[-\pi/2, \pi/2]$ is the trivial solution $\varphi = 0$.

This result can also be obtained via the potential of the weight of (P). According to [11.6], we have

$$\mathcal{V} = -m\vec{g}\cdot\overrightarrow{OG} + const = mg\left(\left(R+\frac{h}{2}\right)\cos\varphi + R\varphi\sin\varphi\right) + const \qquad [11.7]$$

The equilibrium position is given by

$$\frac{\partial\mathcal{V}}{\partial\varphi} = 0 \qquad \leftrightarrow \qquad mg\left[-\frac{h}{2}\sin\varphi + R\varphi\cos\varphi\right] = 0$$

We find the same solution $\varphi = 0$ for equilibrium as above. The stability of the position $\varphi = 0$ is guaranteed if

$$\left.\frac{\partial^2\mathcal{V}}{\partial\varphi^2}\right|_{\varphi=0} = mg\left(R-\frac{h}{2}\right) > 0 \qquad \leftrightarrow \qquad h < 2R$$

11.7.2. *Dynamic analysis*

The kinetic energy is

$$\begin{aligned} E^c_{0P} &= \frac{1}{2}m\vec{V}^2_{0P}(G) + \frac{1}{2}I\dot{\varphi}^2 \qquad \text{where } I \equiv I_P(G\vec{z}_0) = \frac{m}{12}\left(l^2+h^2\right) \\ &= \frac{1}{2}m\left(\frac{h^2}{4}+R^2\varphi^2\right)\dot{\varphi}^2 + \frac{m}{24}\left(l^2+h^2\right)\dot{\varphi}^2 \end{aligned}$$

The potential is still given by [11.7]. Hence, Lagrange's equation is given as

$$\left[m\left(\frac{h^2}{4}+R^2\varphi^2\right)+I\right]\ddot{\varphi} + mR^2\varphi\dot{\varphi}^2 - mg\left(\frac{h}{2}\sin\varphi - R\varphi\cos\varphi\right) = 0$$

11.7.3. *Case of small oscillations*

We now examine the oscillations of (P) around its equilibrium position $\varphi = 0$. The oscillation magnitude around this position are assumed to be an infinitesimal quantity, so that we have the following approximations: $\sin\varphi \approx \varphi$, $\cos\varphi \approx 1$. On the other hand, the terms φ^2 and $\dot{\varphi}^2$ are negligible as they are infinitesimals of second order. The equation of motion then simplifies to

$$\left(mh^2+4I\right)\ddot{\varphi} + mg\left(4R-2h\right)\varphi = 0$$

We derive the period of oscillation around the equilibrium position:

$$T = 2\pi\sqrt{\frac{mh^2+4I}{mg(4R-2h)}}$$

The period is defined only for $R < \frac{h}{2}$. It tends to infinity when $R = \frac{h}{2}$, that is, when the block P is moved away from its equilibrium position $\varphi = 0$, it takes an infinite time to return to this position. In other words, it will not return to this position. We thus indirectly arrive at the result of the previous analysis on the stability around the equilibrium position.

11.8. Disk welded to a rod

Consider a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O;\vec{x}_g,\vec{y}_g,\vec{z}_g)$ with upward vertical $\vec{y}_g$, and a system $\mathcal{S}$ made up of two rigid bodies (T) and (D) moving in the plane $O\vec{x}_g\vec{y}_g$:

– (T) is a homogeneous rod AB with length $4a$ and mass $3m$,

– (D) is a homogeneous disk of center C, radius a and mass $2m$.

We define a basis $(\vec{x}_T, \vec{y}_T, \vec{z}_g)$ attached to the rod (T) such that $\vec{x}_T$ points from A to B, and a basis $(\vec{x}_D, \vec{y}_D, \vec{z}_g)$ attached to the disk (D).

The system $\mathcal{S}$ is in the gravity field r $-g\vec{y}_g$ and is subjected to the following joints:

– the midpoint of (T) coincides with the point O, the pivot joint at O is assumed to be perfect.

– the disk (D) is welded at a point on its circumference to the end B of the rod (T), such that $\vec{x}_D$ is perpendicular to AB and $(\vec{x}_T, \vec{x}_D) = \pi/2$.

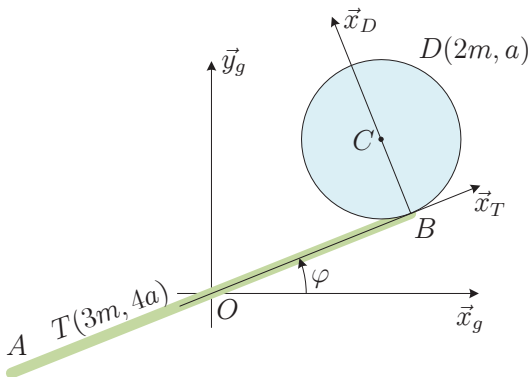


Figure 11.11. *Disk welded to a rod*

This problem will be solved with different parameterizations that yield different information regarding the motions and the constraint inter-efforts at the welded point B.

11.8.1. *First parameterization*

We choose the following parameterization if we wish to study only the equation of motion:

PARAMETERIZATION.

- Primitive parameter: $\varphi \equiv (\vec{x}_g, \overrightarrow{AB})$.
- No primitive constraint equation.
- Retained parameter: φ.
- No complementary constraint equation.

The kinetic energy of the system is

$$2E^c_{g\mathcal{S}} = 15ma^2\dot{\varphi}^2$$

The potential of the weight is

$$2\mathcal{V}_{g\mathcal{S}} = 4mga(2\sin\varphi + \cos\varphi)$$

As the pivot joint at O is perfect, we can apply Lagrange's equation [8.3]:

$$\mathscr{L}_\varphi : \quad \boxed{15ma^2\ddot{\varphi} + 2mga(2\cos\varphi - \sin\varphi) = 0} \qquad [11.8]$$

11.8.2. *First integral*

All the requirements are fulfilled for the energy integral [9.11], $E^c + \mathcal{V} = const$:

$$15ma^2\dot{\varphi}^2 + 4mga(2\sin\varphi + \cos\varphi) = const$$

We can also obtain this first integral by multiplying [11.8] by $2\dot{\varphi}$ and then integrating the result with respect to time.

11.8.3. *Second parameterization*

If there were a perfect pivot joint at the point B instead of a clamp, the radius BC would not be forced to be perpendicular to AB and the disk (D) would rotate relative to the rod (T). The position of the system $\mathcal{S}$ would then be defined by two primitive parameters φ, θ where $\theta \equiv (\vec{x}_g, \overrightarrow{BC})$ (Figure 11.12).

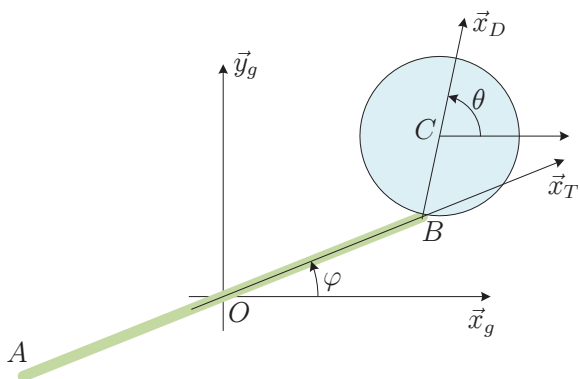


Figure 11.12. *Disk welded to a rod*

The weld at B creates a torque $\Gamma\vec{z}_g$ exerted by (T) on (D) (and the opposite torque exerted by (D) on (T)), which ensures the orthogonality between BC and AB, that is, $\theta = \varphi + \frac{\pi}{2}$. If we wish to know the constraint torque Γ, we must adopt the following parameterization, where relationship $\theta = \varphi + \frac{\pi}{2}$ is classified as complementary (we "release" the weld joint):

PARAMETERIZATION.

- Primitive parameters: φ, θ.
- No primitive constraint equation.
- Retained parameters: φ, θ.
- Complementary constraint equation: $\theta - \varphi = \frac{\pi}{2}$.

The kinetic energies of the rod and the disk are

$$2E^c_{gT} = \frac{1}{3}3m(2a)^2\dot{\psi}^2$$

and

$$2E^c_{gD} = 2m\vec{V}^2_{gD}(C) + \vec{\Omega}_{gD}.\mathcal{J}_D(C)\vec{\Omega}_{gD} \quad \text{with} \quad \begin{cases} \vec{V}_{gD}(C) = 2a\dot{\varphi}\vec{y}_T + a\dot{\theta}\vec{y}_D \\ \vec{\Omega}_{gD}.\mathcal{J}_D(C)\vec{\Omega}_{gD} = \frac{1}{2}2ma^2\dot{\theta}^2 \end{cases}$$

Hence, the kinetic energy of the system:

$$2E^c_{g\mathcal{S}} = ma^2\left[12\dot{\varphi}^2 + 3\dot{\theta}^2 + 8\dot{\varphi}\dot{\theta}\cos(\theta-\varphi)\right]$$

The potential of the weight is

$$2\mathcal{V}_{g\mathcal{S}} = 4mgy_C = 4mga\,(2\sin\varphi + \sin\theta)$$

As the joints in $\mathcal{S}$ are perfect, we can apply Lagrange's equations [8.6] by using λ to denote the Lagrange multiplier:

$$\mathscr{L}_\theta : \quad \boxed{ma^2\left[3\ddot{\theta} + 4\ddot{\varphi}\cos(\theta-\varphi) + 4\dot{\varphi}^2\sin(\theta-\varphi)\right] + 2mgacos\theta = \lambda}$$

$$\mathscr{L}_\varphi : \quad \boxed{ma^2\left[12\ddot{\varphi} + 4\ddot{\theta}\cos(\theta-\varphi) - 4\dot{\theta}^2\sin(\theta-\varphi)\right] + 4mgacos\varphi = -\lambda}$$

Now taking into account the complementary constraint equation, we have

$$\begin{aligned} ma^2(3\ddot{\varphi} + 4\dot{\varphi}^2) - 2mga\sin\varphi &= \lambda \\ ma^2(12\ddot{\varphi} - 4\dot{\varphi}^2) + 4mga\sin\varphi &= -\lambda \end{aligned} \qquad [11.9]$$

By then adding these two equations, we obtain

$$15ma^2\ddot{\varphi} + 2mga\,(2\cos\varphi - \sin\varphi) = 0$$

We thus once again arrive at equation of motion [11.8] obtained using the first parameterization. Once this equation is solved, φ is known as a function of time, and λ can be calculated using either of the two equations [11.9].

To find the physical significance of the multiplier λ, let us calculate the virtual power of the inter-efforts between the rod (T) and the disk (D), which exist at the weld point B:

$$\mathscr{P}^*(\mathcal{F}_{T\leftrightarrow D}) \underset{[5.14]}{=} \vec{R}_{T\to D}.\vec{V}^*_{TD}(B) + \vec{\mathcal{M}}_{T\to D}(B).\vec{\Omega}^*_{TD},$$

where $\vec{R}_{T\to D}$ is the constraint force at B and $\vec{\mathcal{M}}_{T\to D}(B)$ the torque $\Gamma\vec{z}_g$. Further, $\vec{V}^*_{TD}(B) = \vec{V}^*_{gD}(B) - \vec{V}^*_{gT}(B) = \vec{0}$ according to definition [4.11] of the virtual velocity itself, and, since $\vec{\Omega}_{TD} = \vec{\Omega}_{gD} - \vec{\Omega}_{gT} = (\dot{\theta} - \dot{\varphi})\vec{z}_g$, following the procedure described in [4.22] we derive that $\vec{\Omega}^*_{TD} = (\dot{\theta}^* - \dot{\varphi}^*)\vec{z}_g$. Hence,

$$\mathscr{P}^*(\mathcal{F}_{T\leftrightarrow D}) = \Gamma(\dot{\theta}^* - \dot{\varphi}^*) \qquad [11.10]$$

On the other hand, by applying relationship [8.10] to the complementary constraint equation $\theta - \varphi = \frac{\pi}{2}$, we know that the virtual power of the inter-efforts at the weld point B is

$$\mathscr{P}^*(\mathcal{F}_{T\leftrightarrow D}) = \lambda(\dot{\theta}^* - \dot{\varphi}^*) \qquad [11.11]$$

By comparing [11.10] and [11.11], we find $\Gamma = \lambda$: the torque we wished to find is equal to the multiplier.

REMARK. The meaning of the multiplier λ may also be found using Newtonian mechanics, however the calculations involved are longer. Let us write, here, the moment equation according to Newton's laws, which stipulates that the dynamic moment of the disk with respect to the axis $B\vec{z}_g$ is equal to the moment of the external efforts on the disk, with respect to the same axis:

$$\delta_{gD}(B\vec{z}_g) = \mathcal{M}_{\text{ext}\to D}(B\vec{z}_g) \qquad [11.12]$$

The moment of the external efforts on the disk with respect to the axis $B\vec{z}_g$ results from the torque $\Gamma\vec{z}_g$ and the moment of the weight:

$$\mathcal{M}_{\text{ext}\to D}(B\vec{z}_g) = \Gamma + \left[\overrightarrow{BC} \times (-2mg\vec{y}_g)\right].\vec{z}_g = \Gamma + 2mga\sin\varphi \qquad [11.13]$$

The dynamic moment of the disk with respect to the axis $B\vec{z}_g$ is calculated via the center C:

$$\delta_{gD}(B\vec{z}_g) = \delta_{gD}(C\vec{z}_g) + \left[\overrightarrow{BC} \times 2m\vec{\Gamma}_{gD}(C)\right].\vec{z}_g$$

where

– $\delta_{gD}(C\vec{z}_g) = ma^2\ddot{\theta} = ma^2\ddot{\varphi}$

– $\vec{V}_{gD}(C) = 2a\dot{\varphi}\vec{y}_T - a\dot{\varphi}\vec{x}_T \Rightarrow \vec{\Gamma}_{gD}(C) = a\ddot{\varphi}(2\vec{y}_T - \vec{x}_T) - a\dot{\varphi}^2(2\vec{x}_T + \vec{y}_T)$.

Hence

$$\delta_{gD}(B\vec{z}_g) = ma^2(3\ddot{\varphi} + 4\dot{\varphi}^2) \qquad [11.14]$$

With [11.13] and [11.14], relationship [11.12] becomes

$$ma^2(3\ddot{\varphi} + 4\dot{\varphi}^2) = \Gamma + 2mga\sin\varphi$$

By comparing this relationship with $[11.9]_1$, we obtain $\Gamma = \lambda$. ■

11.8.4. *Third parameterization*

The above second parameterization provides the torque at the weld point B. If we now wish to know the complete constraint effort field at B, that is, both the force $\vec{R}_{T\to D}$ and the torque $\vec{\mathcal{M}}_{T\to D}(B) = \Gamma\vec{z}_g$, we must introduce two new parameters, for instance the Cartesian coordinates (x, y) of the point C relative to the coordinate system $(O; \vec{x}_g\vec{y}_g)$ and adopt the new parameterization given below:

PARAMETERIZATION.

- Primitive parameters: x, y, φ, θ.
- No primitive constraint equation.
- Retained parameters: x, y, φ, θ.
- Complementary constraint equations: $x = 2a\cos\varphi + a\sin\theta$, $y = 2a\sin\varphi + a\sin\theta$ and $\theta - \varphi = \frac{\pi}{2}$.

We then obtain four Lagrange's equations and three complementary constraint equations, or a total of seven equations for seven unknowns: four kinematic unknowns x, y, φ, θ and three unknown constraint efforts $\vec{R}_{T\to D}, \Gamma$.

11.9. Motion of two rods

Consider a system $\mathcal{S}$ moving in a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$, with downward vertical $\vec{x}_0$. The system is made up of two homogeneous rods $S_1 = OA$ and $S_2 = AB$, each of which has a mass m and length $2a$. The rod OA is connected at O to the support, and at A to the rod AB, through perfect spherical joints. The rod AB is connected at B to the axis $O\vec{x}_0$ through a ball-and-cylinder joint.

The system $\mathcal{S}$ is moving in the gravity filed $g\vec{x}_0$.

We seek to establish the equations of motion and to study any possible parametric equilibrium positions. For the purposes of this analysis, we introduce the following notations:

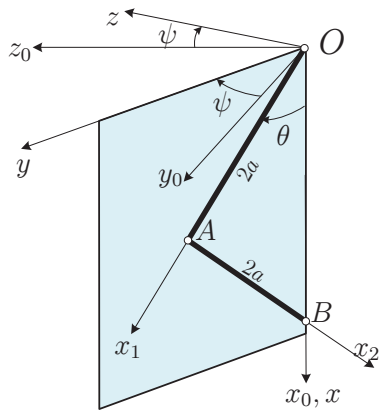


Figure 11.13. *Movement of two rods*

– $O\vec{x}\vec{y}$ the plane of $\mathcal{S}$ (with $\vec{x} = \vec{x}_0$), $\vec{z} \equiv \vec{x} \times \vec{y}$,

– R the rotating reference frame defined by the coordinate system $(O; \vec{x}, \vec{y}, \vec{z})$,

– $\vec{x}_1, \vec{x}_2$: unit vectors orienting $\overrightarrow{OA}, \overrightarrow{AB}$,

– the orthonormal bases $b_1 \equiv (\vec{x}_1, \vec{y}_1 = \vec{z}_1 \times \vec{x}_1, \vec{z}_1 = \vec{z})$, $b_2 \equiv (\vec{x}_2, \vec{y}_2 = \vec{z}_2 \times \vec{x}_2, \vec{z}_2 = \vec{z})$.

We choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: The angles ψ, θ, oriented along $\vec{x}_0$ and $\vec{z}$, respectively, as shown in Figure 11.13.
- No primitive constraint equation.
- Retained parameters: ψ, θ.
- No complementary constraint equation.

11.9.1. *Equations of motion*

– As O is fixed, the kinetic energy of the rod S_1 is

$$E_{01}^c = \frac{1}{2}\vec{\Omega}_{01}.\mathcal{I}_{S_1}(O)\vec{\Omega}_{01} \quad \text{where } \vec{\Omega}_{01} = \dot{\theta}\vec{z} + \dot{\psi}\vec{x} = \dot{\psi}\cos\theta\vec{x}_1 - \dot{\psi}\sin\theta\vec{y}_1 + \dot{\theta}\vec{z}_1$$

$$= \frac{1}{2}{}_{b_1}\langle \dot{\psi}\cos\theta, -\dot{\psi}\sin\theta, \dot{\theta}\rangle \; {}_{b_1}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{m(2a)^2}{3} & 0 \\ 0 & 0 & \frac{m(2a)^2}{3} \end{bmatrix} \cdot {}_{b_1}\begin{Bmatrix} \dot{\psi}\cos\theta \\ -\dot{\psi}\sin\theta \\ \dot{\theta} \end{Bmatrix}$$

$$= \frac{2ma^2}{3}\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right)$$

– The kinetic energy of the rod S_2 is written as

$$E_{02}^c = \frac{1}{2}\vec{\Omega}_{02} \cdot \mathcal{I}_{S_2}(G_2)\vec{\Omega}_{02} + \frac{1}{2}m\vec{V}_{02}^2(G_2)$$

with, consecutively

$$\vec{\Omega}_{02} = -\dot{\theta}\vec{z} + \dot{\psi}\vec{x} = \dot{\psi}\cos\theta\vec{x}_2 + \dot{\psi}\sin\theta\vec{e}_{y_2} - \dot{\theta}\vec{z}_2$$

$$\begin{aligned}\frac{1}{2}\vec{\Omega}_{02}\cdot\mathcal{I}_{S_2}(G_2)\vec{\Omega}_{02} &= \frac{1}{2}{}_{b_2}\langle\dot{\psi}\cos\theta, \dot{\psi}\sin\theta, -\dot{\theta}\rangle_{b_2}\begin{bmatrix}0 & 0 & 0\\ 0 & \frac{m(2a)^2}{12} & 0\\ 0 & 0 & \frac{m(2a)^2}{12}\end{bmatrix}\cdot{}_{b_2}\begin{Bmatrix}\dot{\psi}\cos\theta\\ \dot{\psi}\sin\theta\\ -\dot{\theta}\end{Bmatrix}\\ &= \frac{ma^2}{6}\left(\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\right)\end{aligned}$$

$$\overrightarrow{OG_2} = 3a\cos\theta\vec{x} + a\sin\theta\vec{y}$$

$$\begin{aligned}\vec{V}_{02}(G_2) &= \frac{d_{R_0}\overrightarrow{OG_2}}{dt} \underset{[1.48]}{=} \frac{d_R\overrightarrow{OG_2}}{dt} + \vec{\Omega}_{R_0R}\times\overrightarrow{OG_2}\\ &= -3a\dot{\theta}\sin\theta\vec{x} + a\dot{\theta}\cos\theta\vec{y} + \dot{\psi}\vec{x}\times(3a\cos\theta\vec{x} + a\sin\theta\vec{y})\\ &= -3a\dot{\theta}\sin\theta\vec{x} + a\dot{\theta}\cos\theta\vec{y} + a\dot{\psi}\sin\theta\vec{z}\end{aligned}$$

$$\frac{1}{2}m\vec{V}_{02}^2(G_2) = \frac{1}{2}ma^2\left(\dot{\theta}^2(9\sin^2\theta + \cos^2\theta) + \dot{\psi}^2\sin^2\theta\right)$$

Hence

$$E_{02}^c = \frac{2ma^2}{3}\left[\dot{\psi}^2\sin^2\theta + \dot{\theta}^2\left(1 + 6\sin^2\theta\right)\right]$$

– Finally, the kinetic energy of the system $\mathcal{S}$ is given as

$$E_{0\mathcal{S}}^c = E_{0S_1}^c + E_{0S_2}^c = \frac{4}{3}ma^2\left[\dot{\theta}^2(1 + 3\sin^2\theta) + \dot{\psi}^2\sin^2\theta\right]$$

– As the joints are perfect, what remains to be calculated is the potential of the weight. We have, within an additional constant

$$\begin{aligned}\mathcal{V}_{0\mathcal{S}} &= -m\vec{g}\cdot\overrightarrow{OG_1} - m\vec{g}\cdot\overrightarrow{OG_2}\\ &= -\begin{Bmatrix}mg\\ 0\end{Bmatrix}\cdot\begin{Bmatrix}a\cos\theta\\ a\sin\theta\end{Bmatrix} - \begin{Bmatrix}mg\\ 0\end{Bmatrix}\cdot\begin{Bmatrix}3a\cos\theta\\ a\sin\theta\end{Bmatrix}\\ &= -4mga\cos\theta\end{aligned}$$

Lagrange's equations give the equations of motion:

$$\begin{aligned}\mathscr{L}_\theta :&\quad \frac{2}{3}ma^2\left[\ddot{\theta}(1 + 3\sin^2\theta) + (3\dot{\theta}^2 - \dot{\psi}^2)\sin\theta\cos\theta\right] + mga\sin\theta = 0\\ \mathscr{L}_\psi :&\quad \frac{8}{3}ma^2\left(\ddot{\psi}\sin^2\theta + 2\dot{\psi}\dot{\theta}\sin\theta\cos\theta\right) = 0\end{aligned}$$

When compared to Newtonian mechanics, the second equation corresponds to the conversation of the angular momentum about axis $O\vec{x}_0$.

11.9.2. *Relative equilibrium*

It can be seen that the pair $(\psi, \theta) = (\dot{\psi}_0 t + \psi_0, \theta_0)$, where $\psi_0, \dot{\psi}_0, \theta_0$ are arbitrary initial conditions, may be a solution to the problem. Indeed, by inserting these expressions in the previous equations of motion, we find that equation $\mathscr{L}_\psi$ is identically satisfied, while equation $\mathscr{L}_\theta$ implies that $\dot{\psi}_0$ and θ_0 must satisfy the following condition:

$$\dot{\psi}_0^2 \sin 2\theta_0 = \frac{3g}{a} \sin \theta_0$$

This condition is identically satisfied if $\theta_0 = 0$, which signifies that $\theta = 0$ is a parametric equilibrium regardless of the initial velocity $\dot{\psi}_0$ chosen. If $\theta_0 \neq 0$, then the condition becomes

$$\dot{\psi}_0^2 = \frac{3g}{2a \cos \theta_0}$$

This expression gives the initial velocity $\dot{\psi}_0$ to be imposed on the system in order for $\theta = \theta_0$ to be a parametric equilibrium.

11.10. System with a perfect wire joint

A system $\mathcal{S}$ is moving in a Galilean reference frame $R_g = R_0$ equipped with an orthonormal coordinate system $(O, \vec{x}_g, \vec{y}_g, \vec{z}_g)$ with upward vertical $\vec{z}_g$. This includes

- a rigid body (S) composed of a homogeneous disk with center C, radius a and mass $4m$, to which is welded a homogeneous rod AB of length $2a\sqrt{3}$, mass $3m$, midpoint C and perpendicular to the disk,
- a particle p, whose position in R_0 is P and whose mass is $\frac{7}{4}m$.

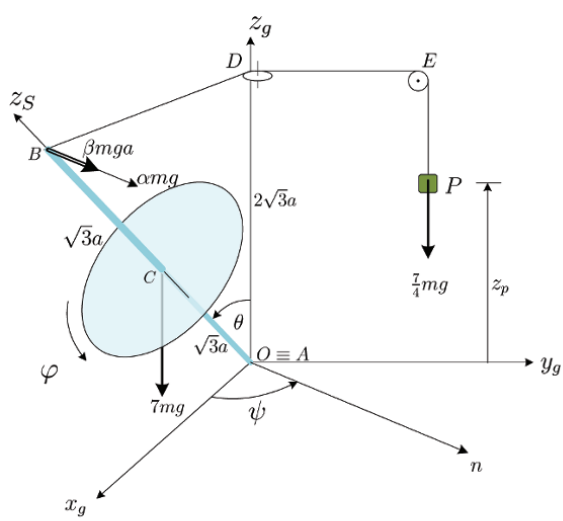


Figure 11.14. *System with a perfect wire joint*

The system $\mathcal{S} = (S) \cup p$ is subjected to the following joints:

– the end A of the rod is attached to O through a perfect spherical joint,

– the end B of the rod is connected to the particle p with a perfect wire (inextensible, massless, perfectly flexible) assumed to be always taut, passing through a ring of negligible dimension, located at the point D such that $\overrightarrow{OD} = 2a\sqrt{3}\vec{z}_g$, and then wrapped over a pulley attached to R_g, having a horizontal axis and negligible mass.

– further, it is assumed that p can only move vertically.

We define the position of (S) by means of the usual Euler angles ψ, θ, φ and we use $\vec{n}$ to denote the unit vector parallel to the line of nodes (refer again to the notations in [2.3]).

The system $\mathcal{S}$ is subjected to the gravity field $-g\vec{z}_g$. In addition, the rigid body (S) is subjected to a set of efforts whose resultant force is $\vec{R} = \alpha mg\vec{n}$ and whose resultant moment at B is $\vec{\Gamma} = \beta mga\vec{n}$, where α, β are constants.

The primitive parameters of the system are the Euler angles ψ, θ, φ for (S) and the elevation z_P for p. The wire joint between (S) and p is expressed by the relationship

$$BD + DE + (OD - z_P) = \text{ the length of the wire, which is constant}$$

which enables us to express z_P as a function of θ:

$$z_P = BD + const = 4\sqrt{3}a \sin\frac{\theta}{2} + const$$

where $const$ denotes a known constant that we do not need to make explicit.

We choose to work with the following parameterization where the previous relationship is classified as a primitive equation:

PARAMETERIZATION.

- Primitive parameters: ψ, θ, φ, z_P.
- Primitive constraint equation: $z_P = 4\sqrt{3}a \sin\frac{\theta}{2} + const$.
- Retained parameters: ψ, θ, φ.
- No complementary constraint equation.

11.10.1. *Lagrange's equations*

Since $\vec{V}_{gS}(O) = \vec{0}$, we have $E^c_{gS} = \frac{1}{2}\vec{\Omega}_{gS}.\mathcal{I}_S(O)\vec{\Omega}_{gS}$. Let us carry out the calculations in the second intermediate basis $v \equiv (\vec{n}, \vec{v}, \vec{z}_S)$ (refer again to the notations in [2.3]), where $\vec{z}_S$ is the unit vector of $\overrightarrow{AB}$:

$$\vec{\Omega}_{gS} = \dot{\theta}\vec{n} + \dot{\psi}\vec{z}_g + \dot{\varphi}\vec{z}_S = \left\{ \begin{array}{c} \dot{\theta} \\ \dot{\psi}\sin\theta \\ \dot{\varphi} + \dot{\psi}\cos\theta \end{array} \right\}_v$$

and the inertia matrix of (S), at O and in the basis v, is written as follows, with $[C, 4m]$ denoting the fictitious particle of position C and mass $4m$:

$$\begin{aligned}
\mathbb{I}_S(O;v) &= \mathbb{I}_{\text{stem}}(O;v) + \mathbb{I}_{[C,4m]}(O,v) + \mathbb{I}_{\text{disk}}(C;v) \\
&= 12ma^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix} + 4ma^2 \begin{pmatrix} 3 & & \\ & 3 & \\ & & 0 \end{pmatrix} + ma^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 2 \end{pmatrix} \\
&= ma^2 \begin{pmatrix} 25 & & \\ & 25 & \\ & & 2 \end{pmatrix}
\end{aligned}$$

Hence the kinetic energy of (S) is given as:

$$E^c_{gS} = \frac{1}{2}ma^2 \left[25\dot{\theta}^2 + 25\dot{\psi}^2 \sin^2\theta + 2(\dot{\varphi} + \dot{\psi}\cos\theta)^2\right]$$

The kinetic energy of the particle is given as

$$E^c_{gp} = \frac{1}{2}\,\frac{7}{4}m\dot{z}_P^2 = \frac{21}{2}ma^2\cos^2\frac{\theta}{2}\dot{\theta}^2$$

It finally gives us

$$E^c_{g\mathscr{S}} = \frac{1}{2}ma^2 \left[\dot{\theta}^2\left(25 + 21\cos^2\frac{\theta}{2}\right) + 25\dot{\psi}^2\sin^2\theta + 2(\dot{\varphi} + \dot{\psi}\cos\theta)^2\right]$$

The potential of the weight is given as

$$\mathcal{V} = 7m\,gz_C + \frac{7}{4}m\,gz_P + const = 7\sqrt{3}mga\left(\cos\theta + \sin\frac{\theta}{2}\right) + const$$

Let us now calculate the power of efforts other than the weight. These efforts are

– the efforts internal to the system $\mathcal{S}$, namely

1. the efforts within the rigid body (S),
2. the constraint efforts due to the wire,

– the external efforts that are not derivable from a potential, which are

3. the constraint efforts at O,
4. the given efforts applied to (S), whose resultant force is $\vec{R} = \alpha mg\vec{n}$ and whose resultant moment at B is $\vec{\Gamma} = \beta mga\vec{n}$.

Let us examine, in order, the VPs of these efforts:

1. The VP of the efforts within the rigid body (S) is zero according to [5.2].
2. As the wire is perfect and as the pulley has no mass, the wire joint is perfect. As there is no complementary constraint equation, the VP of the constraint efforts due the wire joint is zero.

   REMARK. Let us verify through a direct calculation that the VP of the constraint efforts due to the wire is zero. Let T denote the tension in the wire and $\vec{i}$ denote the unit vector orienting $\overrightarrow{BD}$; this VP is written as

$$\mathscr{P}^*_{R_g}(\mathcal{F}_{\text{wire}\to\mathcal{S}}) = T\vec{i}.\vec{V}^*_{gS}(B) + T\vec{z}_g.\vec{V}^*_g(p)$$

where, on the one hand

$$\vec{V}^*_{gS}(B) = \underbrace{\vec{V}^*_{gS}(A)}_{\vec{0}} + \vec{\Omega}^*_{gS} \times \overrightarrow{AB} = 2\sqrt{3}a(\dot{\psi}^* \sin\theta \vec{n} - \dot{\theta}^* \vec{v})$$

$$BD = 4\sqrt{3}a \sin\frac{\theta}{2} \qquad \Rightarrow \qquad \vec{i} = \frac{\overrightarrow{BD}}{BD} = \frac{\overrightarrow{OD} - \overrightarrow{OB}}{BD} = \frac{1}{2\sin\frac{\theta}{2}}(\vec{z}_g - \vec{z}_S)$$

Hence $\vec{i}.\vec{V}^*_{gS}(B) = -2\sqrt{3}a\cos\frac{\theta}{2}\dot{\theta}^*$. On the other hand

$$\vec{V}^*_g(p) = \frac{\partial\overrightarrow{OP}}{\partial\theta}\dot{\theta}^* \qquad \Rightarrow \qquad \vec{z}_g.\vec{V}^*_g(p) = \vec{z}_g.\frac{\partial\overrightarrow{OP}}{\partial\theta}\dot{\theta}^* = \frac{\partial z_P}{\partial\theta}\dot{\theta}^* = 2\sqrt{3}a\cos\frac{\theta}{2}\dot{\theta}^*$$

We, thus, do indeed obtain $\mathscr{P}^*_{R_g}(\mathcal{F}_{\text{wire}\to S}) = 0$. ■

3. As the joint at O is perfect and there is no complementary constraint equation present, the VP of the constraint efforts at O is zero.

4. The VP of the given efforts applied on (S) is $\mathscr{P}^* = \vec{R}.\vec{V}^*_{gS}(B) + \vec{\Gamma}.\vec{\Omega}^*_{gS}$, where $\vec{\Omega}^*_{gS} = \dot{\theta}^*\vec{n} + \dot{\psi}^*\vec{z}_g + \dot{\varphi}^*\vec{z}_S$ and $\vec{V}^*_{gS}(B)$ has been obtained earlier. Hence

$$\mathscr{P}^* = mga(2\sqrt{3}\alpha\sin\theta\dot{\psi}^* + \beta\dot{\theta}^*)$$

Using the previous results, we obtain the Lagrange equations:

$$\mathscr{L}_\varphi: \quad \frac{d}{dt}(\dot{\varphi} + \dot{\psi}\cos\theta) = 0 \quad \rightarrow \quad \dot{\varphi} + \dot{\psi}\cos\theta = r_0 \quad \text{where} \quad r_0 = \dot{\varphi}_0 + \dot{\psi}_0\cos\theta_0 = const$$

$$\mathscr{L}_\psi: \quad \frac{d}{dt}(25\dot{\psi}\sin^2\theta + 2r_0\cos\theta) = 2\sqrt{3}\alpha\omega_0^2\sin\theta \qquad \text{with} \quad \omega_0^2 \equiv \frac{g}{a}$$

$$\mathscr{L}_\theta: \quad \frac{d}{dt}\left[\dot{\theta}\left(25 + 21\cos^2\frac{\theta}{2}\right)\right] + \frac{21}{2}\dot{\theta}^2\sin\frac{\theta}{2}\cos\frac{\theta}{2} + \dot{\psi}\sin\theta(2r_0 - 25\dot{\psi}\cos\theta)$$
$$+ 7\sqrt{3}\omega_0^2\left(-\sin\theta + \frac{1}{2}\cos\frac{\theta}{2}\right) = \beta\omega_0^2$$

11.10.2. *First integral*

Equation $\mathscr{L}_\varphi$ is a first integral. It may also be obtained using Newtonian mechanics as follows:

– it can be easily verified that the moment of the external efforts on S about axis $O\vec{z}_S$ is zero: $\mathcal{M}_{ext\to S}(O\vec{z}_S) = 0$,

– furthermore, the point O is fixed in both R_g and (S), the vector $\vec{z}_S$ is fixed in (the vector space defined by) (S), and finally the inertia operator $\mathcal{I}_S(O)$ of (S) about point O is axisymmetric with respect to the axis $O\vec{z}_S$ and the moment of inertia $I_S(O\vec{z}_S)$ of (S) about the same axis is non-zero.

As a result, Euler's first integral exists for the rigid body (S), which is written exactly like the previous equation $\mathscr{L}_\varphi$.

11.11. Rotating disk–rod system

Consider a system $\mathcal{S}$ moving in a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_g, \vec{y}_g, \vec{z}_g)$. This system is composed of two rigid bodies:

– a homogeneous rod $(T) = (AC)$ with length $2a$, mass $3m$ and center of mass G,

– a homogeneous disk (D) with radius a, mass $4m$ and center of mass C.

The rod (T) is horizontal, connected to the axis $O\vec{z}_g$ through a perfect cylindrical joint. We define the orthonormal basis $(\vec{n}, \vec{u}, \vec{z}_g)$ attached to (T) such that $\vec{n}$ points from A to C and let ψ denotes the angle $(\vec{x}_g, \vec{n})$.

The perfect joint at the point C between the rod (T) and the disk (D) allows

– on the one hand, (D) to swivel about $\vec{z}_g$, by the angle θ measured around the axis $C\vec{n}$,

– on the other hand, (D) to spin around its axis of revolution $C\vec{z}_S$ by the angle φ.

The angles ψ, θ, φ that are thus defined are the Euler angles that allows us to define the position (D) relative to R_g. The second intermediate Euler basis is denoted by $v \equiv (\vec{n}, \vec{v}, \vec{z}_S)$ (refer again to the notations [2.3]).

The disk (D) is also in frictionless contact at the point I with the horizontal plane $O\vec{x}_g\vec{y}_g$.

The whole system $\mathcal{S} = (T) \cup (D)$ is subjected to gravity $-g\vec{z}_g$.

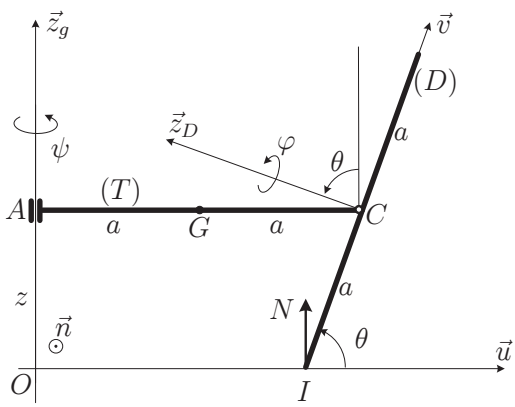


Figure 11.15. *Rotating disk–rod system*

We will solve this problem using two different parameterizations.

11.11.1. *Independent parameterization*

We first choose to work with the following parameterization:

PARAMETERIZATION.

- Primitive parameters: $z \equiv \overrightarrow{OA}.\vec{z}_g$, ψ, θ, φ.
- Primitive constraint equation: $z = a \sin\theta$, which expresses the contact at I.
- Retained parameters: ψ, θ, φ.
- No complementary constraint equation.

The kinetic energy of the rod (T) is $E^c_{gT} = \frac{1}{2}3m\vec{V}^2_{gT}(G) + \frac{1}{2}\vec{\Omega}_{gT}.\mathcal{I}_T(G)\vec{\Omega}_{gT}$, where

$$\vec{V}_{gT}(G) = \dot{z}\vec{z}_0 - a\dot{\psi}\vec{n} = a\dot{\theta}\cos\theta\vec{z}_0 - a\dot{\psi}\vec{n}$$

Hence

$$E^c_{gT} = \frac{1}{2}ma^2(4\dot{\psi}^2 + 3\dot{\theta}\cos^2\theta)$$

The kinetic energy of the disk (D) is $E^c_{gD} = \frac{1}{2}4m\vec{V}^2_{gD}(C) + \frac{1}{2}\vec{\Omega}_{gD}.\mathcal{I}_D(C)\vec{\Omega}_{gD}$, hence

$$\vec{V}_{gD}(C) = \dot{z}\vec{z}_0 - 2a\dot{\psi}\vec{n} = a\dot{\theta}\cos\theta\vec{z}_0 - 2a\dot{\psi}\vec{n} \qquad \text{and} \qquad \mathbb{I}_D(C;v) = ma^2\begin{pmatrix}1 & & \\ & 1 & \\ & & 2\end{pmatrix}$$

Hence

$$E^c_{gD} = \frac{1}{2}ma^2\left[\dot{\theta}^2(1 + 4\cos\cos^2\theta) + \dot{\psi}^2(16 + \sin^2\theta) + 2r^2\right] \qquad \text{where } r \equiv \dot{\varphi} + \dot{\psi}\cos\theta$$

Hence, the kinetic energy of the system is given as

$$E^c_{gS} = \frac{1}{2}ma^2\left[\dot{\theta}^2(1 + 7\cos^2\theta) + \dot{\psi}^2(20 + \sin^2\theta) + 2r^2\right] \qquad [11.15]$$

The potential of the weight is written as

$$\mathcal{V} = 7mgz + const = 7mga\sin\theta + const \qquad [11.16]$$

Hence Lagrange's equations, by denoting $\omega_0^2 \equiv \frac{g}{a}$:

$$\mathscr{L}_\psi : \quad \boxed{\dot{\psi}(20 + \sin^2\theta) + 2r\cos\theta = const} \qquad [11.17]$$

$$\mathscr{L}_\theta : \quad \boxed{\ddot{\theta}(1 + 7\cos^2\theta) - 7\dot{\theta}^2\sin\theta\cos\theta - \dot{\psi}^2\sin\theta\cos\theta + 2r\dot{\psi}\sin\theta + 7\omega_0^2\cos\theta = 0} \qquad [11.18]$$

$$\mathscr{L}_\varphi : \quad \boxed{r = r_0} \qquad [11.19]$$

By comparison with Newtonian mechanics, equation $\mathscr{L}_\varphi$ is the Euler first integral of the disk, equation $\mathscr{L}_\psi$ is the first integral of the conservation of the angular momentum of the system about the axis $A\vec{z}_g$ and equation $\mathscr{L}_\theta$ is equivalent to the energy first integral.

11.11.2. *Total parameterization*

The above parameterization does not enable us to access the contact force at I applied to (D), which has the form $\vec{R}_{\text{support}\to D} = N\vec{z}_g$. To know this force, we must adopt the following parameterization where the contact relationship at I, $z = a\sin\theta$, is classified as a complementary equation:

PARAMETERIZATION.

- Primitive parameters: z, ψ, θ, φ.
- No primitive constraint equation.
- Retained parameters: z, ψ, θ, φ.
- Complementary constraint equation: $z = a\sin\theta$.

The same calculations as above give

$$E_{gT}^{c} = \frac{1}{2}4ma^2\dot{\psi}^2 + \frac{1}{2}3m\dot{z}^2$$

$$E_{gD}^{c} = \frac{1}{2}ma^2\left[\dot{\theta}^2 + \dot{\psi}^2(16 + \sin^2\theta) + 2r^2\right] + \frac{1}{2}4m\dot{z}^2$$

Hence the kinetic energy of the system is given as

$$E_{0\mathscr{S}}^{c} = \frac{1}{2}ma^2\left[\dot{\theta}^2 + \dot{\psi}^2(20 + \sin^2\theta) + 2r^2\right] + \frac{1}{2}7m\dot{z}^2$$

The potential of the weight is written as

$$\mathcal{V} = 7mgz + const$$

As all joints in $\mathcal{S}$ are perfect, we can apply Lagrange's equations [8.6] with the Lagrange multiplier denoted by λ:

$$\begin{aligned}
&\mathscr{L}_z: && 7m\ddot{z} + 7mg = \lambda \\
&\mathscr{L}_\psi: && \dot{\psi}(20 + \sin^2\theta) + 2r\cos\theta = const \qquad \text{identical to [11.17]} \\
&\mathscr{L}_\theta: && ma^2\left(\ddot{\theta} + \dot{\psi}^2\sin\theta\cos\theta + 2r\dot{\psi}\sin\theta\right) = -a\lambda\cos\theta \\
&\mathscr{L}_\varphi: && r = r_0 \qquad \text{identical to [11.19]}
\end{aligned}$$

Now taking into account the complementary constraint equation $z = a\sin\theta$ and after some rearrangement, we arrive at the same system [11.17–11.19], plus an additional relationship for λ:

$$\boxed{\lambda = 7mg + +7ma(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta)}$$

$$\boxed{\dot{\psi}(20 + \sin^2\theta) + 2r_0\cos\theta = const}$$

$$\boxed{\ddot{\theta}(1 + 7\cos^2\theta) - 7\dot{\theta}^2\sin\theta\cos\theta - \dot{\psi}^2\sin\theta\cos\theta + 2r\dot{\psi}\sin\theta + 7\omega_0^2\cos\theta = 0}$$

$$\boxed{r = r_0}$$

Knowing the initial conditions, solving these four equations yields the kinematic unknowns ψ, θ, φ as well as the multiplier λ as a function of time.

Let us show that the Lagrange multiplier λ is equal to the normal contact force N that we wish to find. Indeed, by applying relationship [8.10] to the complementary constraint equation $z = a\sin\theta$, we know that the virtual power of the constraint efforts $\mathcal{F}_{\text{support}\to D}$ is

$$\mathscr{P}^*(\mathcal{F}_{\text{support}\to D}) = \lambda(\dot{z}^* - a\cos\theta\dot{\theta}^*) \qquad [11.20]$$

On the other hand, from the very definition of the virtual power of the contact force at I, we have

$$\mathscr{P}^*(\mathcal{F}_{\text{support}\to D}) = \vec{R}_{\text{support}\to D} \cdot \vec{V}_{gD}^*(I)$$

where

$$\vec{V}_{gD}^*(I) = \vec{V}_{gD}^*(C) + \vec{\Omega}_{gD}^* \times \overrightarrow{CI} = \dot{z}^*\vec{z}_g - 2a\dot{\psi}^*\vec{n} - a\dot{\theta}^*\vec{z}_D + a(\dot{\psi}^*\cos\theta + \dot{\varphi}^*)\vec{n}$$

Hence

$$\mathscr{P}^*(\mathcal{F}_{\text{support}\to D}) = N(\dot{z}^* - a\cos\theta\dot{\theta}^*) \qquad [11.21]$$

By comparing [11.20] and [11.21], we do indeed have $\lambda = N$.

11.11.3. *Engine torque*

It is now assumed that an engine torque $\vec{\Gamma} = \Gamma \vec{z}_0$ is applied by the support on (T) to enforce a constant angular velocity ω onto (T): $\dot{\psi} = \omega = const$. To determine this torque, we choose the following parameterization, where the constraint equation $\dot{\psi} = \omega$ is counted as a complementary equation:

PARAMETERIZATION.

- Primitive parameters: ψ, θ, φ.
- No primitive constraint equation.
- Retained parameters: ψ, θ, φ.
- Complementary constraint equation: $\dot{\psi} = \omega$.

The kinetic energy of the system and the potential of the weight have the same expressions as [11.15] and [11.16]. Lagrange's equations [8.6] give, with λ denoting the Lagrange multiplier:

$$\mathscr{L}_\psi : \quad \boxed{ma^2 \frac{d}{dt}\left[\dot{\psi}\left(20 + \sin^2\theta\right) + 2r\cos\theta\right] = \lambda}$$

$$\mathscr{L}_\theta, \mathscr{L}_\varphi : \quad \text{identical to [11.18] and [11.19]}$$

Now taking into account $\dot{\psi} = \omega$, we obtain

$$\boxed{\lambda = -2ma^2\dot{\theta}\dot{\varphi}\sin\theta = -2ma^2\dot{\theta}\sin\theta\left(r_0 - \omega\cos\theta\right)}$$

As the virtual power of the engine torque is, by definition, $\mathscr{P}^*\left(\mathcal{F}_{\text{engine}\to T}\right) = \Gamma\dot{\psi}^*$, we can immediately verify that the multiplier λ is merely equal to the desired engine torque Γ.

11.12. Dumbbell

A system $\mathcal{S}$ moving in a Galilean reference frame $R_g = R_0$ endowed with an orthonormal coordinate system $(O; \vec{x}_g, \vec{y}_g, \vec{z}_g)$, with $\vec{z}_g$ being upward vertical. It is made up of two rigid bodies S_1 and S_2:

– S_1 is a homogeneous rod AB with length $3a/\sqrt{7}$ and mass m.

– S_2 takes the form of a dumbbell made up of a massless rod CD with length $2a$ and of two identical, homogeneous disks, with radius a, mass m, and respective centers C and D. The two disks are welded at the ends of the rod CD, perpendicularly to CD.

The solids are subjected to the following joints:

– S_1 is connected to the support through a perfect pivot joint whose axis is $O\vec{z}_g$. The end A coincides with O and the angle of rotation of S_1 about $O\vec{z}_g$ is $\psi = (\vec{x}_g, \vec{n})$, where $\vec{n}$ is the unit vector orienting $\overrightarrow{AB}$.

– The midpoint of CD coincides with B. S_2 is connected to S_1 through a system of two perfect pivots:

  - one of axis $O\vec{n}$ that allows the rotation $\theta = (\vec{z}_g, \vec{z}_{S_2})$ of S_2 about $O\vec{n}$, where $\vec{z}_{S_2}$ is the unit vector orienting $\overrightarrow{CD}$.

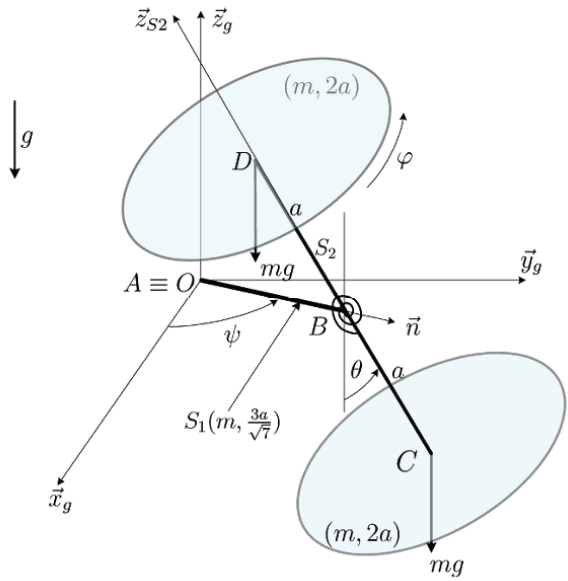


Figure 11.16. *Dumbbell*

- the other of axis $B\vec{z}_{S_2}$, that allows the spin φ of S_2 around $B\vec{z}_{S_2}$.

– A torsion spring with negligible mass, stiffness C, axis $B\vec{n}$, whose one end is attached to S_1 while the other end is attached to CD, is opposed to the θ rotation. This spring is unstretched when $\theta = 0$.

The whole system is subjected to gravity $-g\vec{z}_g$.

The chosen parameterization is as follows:

PARAMETERIZATION.

- Primitive parameters: the angles ψ, θ, φ, which are the usual Euler angles, ψ defining the position of S_1 and, in addition, θ, φ defining the position of S_2.
- No primitive constraint equation.
- Retained parameters: ψ, θ, φ.
- No complementary constraint equation.

11.12.1. *Equations set*

The kinetic energy of S_1 is simple:

$$E_{g1}^c = \frac{1}{2}\frac{1}{3}m\left(\frac{3a}{\sqrt{7}}\right)^2\dot{\psi}^2$$

The kinetic energy of S_2 is obtained by

$$E^c_{g2} = \frac{1}{2}\, 2m \underbrace{\vec{V}^2_{g2}(B)}_{9/7\,\dot{\psi}^2} + \frac{1}{2}\vec{\Omega}_{g2}.\mathcal{I}_{S_2}(B)\vec{\Omega}_{g2}$$

Let us calculate the inertia matrix of S_2 about B, written in the basis $v \equiv (\vec{n}, \vec{v}, \vec{z}_{S_2})$:

$$\mathbb{I}_{S_2}(B; v) = \mathbb{I}_{\text{disk with center } D}(B; v) + \mathbb{I}_{\text{disk with center } C}(B; v)$$

where, by denoting by $[D, m]$ the fictitious particle located at D and of mass m:

$$\begin{aligned}
\mathbb{I}_{\text{disk with center } D}(B; v) &= \mathbb{I}_{[D,m]}(B; v) + \mathbb{I}_{\text{disk with center } D}(D; v) \\
&= \begin{pmatrix} ma^2 & & \\ & ma^2 & \\ & & 0 \end{pmatrix} + \begin{pmatrix} \frac{m}{4}(2a)^2 & & \\ & \frac{m}{4}(2a)^2 & \\ & & \frac{m}{2}(2a)^2 \end{pmatrix} \\
&= ma^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}
\end{aligned}$$

As the inertia matrix $\mathbb{I}_{\text{disk with center } C}(B; v)$ of the disk with center C has the same expression, the inertia matrix of S_2 is

$$\mathbb{I}_{S_2}(B; v) = 4ma^2 \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Finally, the kinetic energy of the system is

$$E^c_{g\mathcal{S}} = \frac{1}{2}ma^2 \left[4\dot{\theta}^2 + (3 + 4\sin^2\theta)\dot{\psi}^2 + 4(\dot{\varphi} + \dot{\psi}\cos\theta)^2\right] \qquad [11.22]$$

As the elevations z_{G_1}, z_{G_2} of the centers of S_1, S_2 are zero, the potential of the efforts is reduced to that of the torsion spring

$$\mathcal{V} = \frac{1}{2}C\theta^2 + const$$

Lagrange's equations are

$$\mathscr{L}_\varphi : \quad \boxed{\dot{\varphi} + \dot{\psi}\cos\theta = r_0} \qquad \text{where} \quad r_0 = \dot{\varphi}_0 + \dot{\psi}_0\cos\theta_0 = const \qquad [11.23]$$

$$\mathscr{L}_\theta : \quad \ddot{\theta} + \dot{\varphi}\dot{\psi}\sin\theta + \frac{\omega_0^2}{4}\theta = 0 \qquad \text{with} \quad \omega_0^2 \equiv \frac{C}{ma^2} \qquad [11.24]$$

$$\mathscr{L}_\psi : \quad \boxed{\left(3 + 4\sin^2\theta\right)\dot{\psi} + 4r_0\cos\theta = K} \qquad \text{where } K = const \qquad [11.25]$$

11.12.2. *First integrals*

The equations [11.23] and [11.25] are first integrals. They may be obtained again using Newtonian mechanics: it can be verified that the first integral is the Euler first integral and that the second comes from the conservation of the angular momentum of the system $\mathcal{S}$ with respect to axis $O\vec{z}_g$.

Furthermore, all the hypotheses of the energy integral [9.11], $E^c + \mathcal{V} = const$, are satisfied here. We thus have

$$4\dot{\theta}^2 + (3 + 4\sin^2\theta)\dot{\psi}^2 + 4(\dot{\varphi} + \dot{\psi}\cos\theta)^2 + \omega_0^2\theta^2 = const \qquad [11.26]$$

Let us use [11.23] and [11.25] to eliminate $\dot{\psi}, \dot{\varphi}$ from [11.26]:

$$\dot{\varphi} + \dot{\psi}\cos\theta = r_0 \qquad \dot{\psi} = \frac{K - 4r_0\cos\theta}{3 + 4\sin^2\theta} = \frac{K - 4r_0\cos\theta}{7 - 4\cos^2\theta}$$

Inserting these expressions in [11.26] leads to the so-called equation for the principal parameter θ:

$$\boxed{4\dot{\theta}^2 = h - \omega_0^2\theta^2 - \frac{(K - 4r_0\cos\theta)^2}{7 - 4\cos^2\theta}} \qquad (h = const) \qquad [11.27]$$

Below is another way to obtain the above-found first integral. Let us again use [11.23] and [11.25] to eliminate $\dot{\psi}, \dot{\varphi}$ from [11.24]:

$$\dot{\psi} = \frac{K - 4r_0\cos\theta}{3 + 4\sin^2\theta} = \frac{K - 4r_0\cos\theta}{7 - 4\cos^2\theta} \qquad \dot{\varphi} = \frac{7r_0 - K\cos\theta}{7 - 4\cos^2\theta} \qquad [11.28]$$

By inserting these expressions in [11.24], we obtain a differential equation in θ:

$$\ddot{\theta} + \frac{(7r_0 - K\cos\theta)(K - 4r_0\cos\theta)}{(7 - 4\cos^2\theta)^2}\sin\theta + \frac{\omega_0^2}{4}\theta = 0 \qquad [11.29]$$

By multiplying this relationship by $\dot{\theta}$ and then by integrating the result with respect to time, we find [11.27].

REMARK. Let us obtain the previous first integrals by means of Newtonian mechanics.

1. Equation [11.23] is the Euler first integral for the rigid body (S_2). Indeed:
   (a) the moment of the external efforts on S_2 about axis $B\vec{z}_{S_2}$ is zero: $\mathcal{M}_{ext\to S_2}(B\vec{z}_{S_2}) = 0$,
   (b) further, the point B is the center of mass of (S_2), the vector $\vec{z}_{S_2}$ is attached to (S_2), and finally the inertia operator $\mathcal{I}_{S_2}(B)$ of (S_2) about B is axisymmetric with respect to the axis $B\vec{z}_{S_2}$ and the moment of inertia $I_{S_2}(O\vec{z}_{S_2})$ of (S_2) with respect to the same axis is non-zero.

2. Further, as the moment $\mathcal{M}_{ext\to\mathcal{S}}(O\vec{z}_g)$ of the external efforts on $\mathcal{S}$, with respect to the fixed axis $O\vec{z}_g$, is zero we have the first integral of the angular momentum: $\sigma_{g\mathcal{S}}(O\vec{z}_g) = 0$. Let us determine the explicit expression for the angular momentum in question:

$$\sigma_{g\mathcal{S}}(O\vec{z}_g) = \sigma_{gS_1}(O\vec{z}_g) + \sigma_{gS_2}(O\vec{z}_g)$$

   where

$$\sigma_{gS_1}(O\vec{z}_g) = \frac{1}{2}m\left(\frac{3a}{\sqrt{7}}\right)^2\dot{\psi} = \frac{3}{7}ma^2\dot{\psi}$$

   and, after lengthy calculations:

$$\sigma_{gS_2}(O\vec{z}_g) = \sigma_{gS_2}(B\vec{z}_g) + \vec{z}_g.\left(\overrightarrow{OB} \times 2m\vec{V}_{g2}(B)\right) = 4ma^2(\dot{\psi} + \dot{\varphi}\cos\theta) + \frac{18}{7}ma^2\dot{\psi}$$

   The equality $\sigma_{g\mathcal{S}}(O\vec{z}_g) = 0$ thus gives

$$7\dot{\psi} + 4\dot{\varphi}\cos\theta = const$$

   Taking into account $\dot{\varphi} + \dot{\psi}\cos\theta = r_0$, we find [11.25].

3. Finally, the external and internal efforts on the system $\mathcal{S}$ do no work or are derivable from a potential. We thus have the energy integral $E^c_{g\mathcal{S}} + \mathcal{V} = const$, whose expression is exactly [11.26] and which also leads to [11.27]. ■

11.12.3. *Analysis with specific initial conditions*

We study the system by taking the following initial conditions:

$$\theta_0 = \frac{\pi}{2} \qquad \dot{\theta}_0 = 0 \qquad \dot{\varphi}_0 = -\frac{\dot{\psi}_0}{4} \tag{11.30}$$

These values provide the constants r_0, K and h :

$$r_0 = \dot{\varphi}_0 = -\frac{\dot{\psi}_0}{4} \qquad K = 7\dot{\psi}_0 \qquad h = \omega_0^2 \frac{\pi^2}{4} + 7\dot{\psi}_0^2$$

Equations [11.27] and [11.29] then become

$$4\dot{\theta}^2 = \left(\frac{\pi\omega_0}{2}\right)^2 + 7\dot{\psi}_0^2 - \omega_0^2\theta^2 - \frac{(7+\cos\theta)^2}{7-4\cos^2\theta}\dot{\psi}_0^2 \tag{11.31}$$

$$\ddot{\theta} - 7\dot{\psi}_0 \frac{\left(\frac{1}{4}+\cos\theta\right)\left(7\dot{\psi}_0 + \dot{\psi}_0\cos\theta\right)}{\left(7-4\cos^2\theta\right)^2}\sin\theta + \frac{\omega_0^2}{4}\theta = 0 \tag{11.32}$$

11.12.4. *Relative equilibrium*

Let us assume that there exists a parametric equilibrium $\theta = \theta_0$. According to [11.28], as θ is constant, the velocities $\dot{\psi}$ and $\dot{\varphi}$ are also constant:

$$\dot{\psi} = \dot{\psi}_0 = const \qquad \dot{\varphi} = \dot{\varphi}_0 = const$$

Using the same initial conditions [11.30], let us find the initial value $\dot{\psi}_0$ for $\theta = \frac{\pi}{2}$ to be a parametric equilibrium. By making $\theta = \frac{\pi}{2}$ in [11.32], we obtain the initial velocity $\dot{\psi}_0$ which must be imposed upon the system:

$$\dot{\psi}_0^2 = \omega_0^2 \frac{\pi}{2} \qquad \Leftrightarrow \qquad \dot{\psi}_0 = \pm\omega_0\sqrt{\frac{\pi}{2}}$$

Note that the first integral [11.31] is identically satisfied with $\theta = \frac{\pi}{2}$ and gives no information on $\dot{\psi}_0$.

11.13. Dumbbell under engine torque

Consider the same system as in the previous example. The only difference here is that we remove the torsion spring at B and apply a torque $\vec{\Gamma} = \Gamma\vec{z}_g$ on S_1 by means of an engine whose stator is fixed in R_g. This torque imposes a constant angular velocity $\omega > 0$ about the axis $O\vec{z}_g$:

$$\dot{\psi} = \omega \quad \leftrightarrow \quad \psi = \omega t + \psi_0$$

We will solve this problem using two different parameterizations: the first gives only the equations of the motion, and the second provides, additionally, access to the engine torque that ensures the constraint $\dot{\psi} = \omega$.

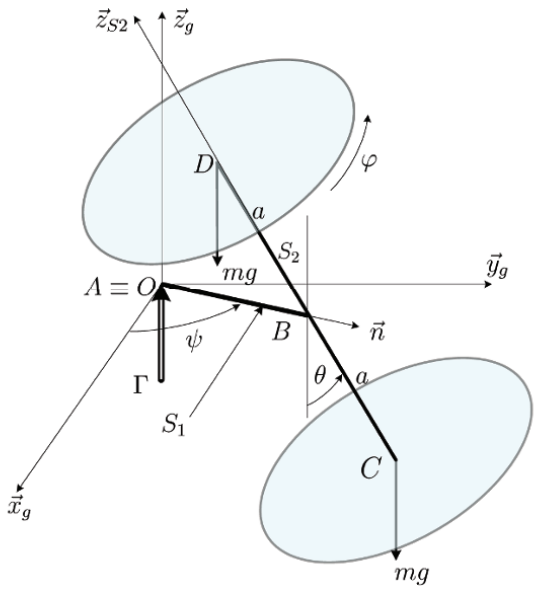


Figure 11.17. *Dumbbell under engine torque*

11.13.1. *Independent parameterization*

PARAMETERIZATION.

- Primitive parameters: ψ, θ, φ.
- Primitive constraint equation: $\psi = \omega t + \psi_0$.
- Retained parameters: θ, φ.
- No complementary constraint equations.

The kinetic energy of the system is given by [11.22] by making $\dot{\psi} = \omega$:

$$E^c_{g\mathscr{S}} = \frac{1}{2} m a^2 \left[4\dot{\theta}^2 + (3 + 4\sin^2\theta)\omega^2 + 4(\dot{\varphi} + \omega\cos\theta)^2 \right] \qquad [11.33]$$

As the elevations z_{G_1}, z_{G_2} of the centers of S_1, S_2 are zero, the potential due to the weight is zero. The virtual power of the engine torque $\vec{\Gamma} = \Gamma \vec{z}_g$ is

$$\mathscr{P}^*(\vec{F}_{\text{engine}\to S_1}) = \vec{\Gamma} \cdot \vec{\Omega}^*_{g1}$$

Further, since $\vec{\Omega}_{g1} = \omega \vec{z}_g$, we have $\vec{\Omega}^*_{g1} = \vec{0}$ (refer again to the procedure described in [4.22]). Hence, $\mathscr{P}^*(\vec{F}_{\text{engine}\to S_1}) = 0$!

Lagrange's equations are given as

$$\mathscr{L}_\varphi : \quad \frac{\partial E^c}{\partial \dot{\varphi}} = const \quad \rightarrow \quad \boxed{\dot{\varphi} + \omega \cos\theta = r_0}$$

$$\mathscr{L}_\theta : \quad ma^2 \left[4\ddot{\theta} - 4\sin\theta\cos\theta\omega^2 + 4\left(\dot{\varphi} + \omega\cos\theta\right)\omega\cos\theta\right] = 0$$

$$\rightarrow \quad \boxed{\ddot{\theta} - \sin\theta\cos\theta\omega^2 + r_0\omega\cos\theta = 0}$$

Equation $\mathscr{L}_\varphi$ thus gives a first integral. By multiplying the second equation by 2θ and then integrating the result with respect to time, we arrive at another first integral, which involves the unknown θ only (equation for the principal parameter):

$$\boxed{\dot{\theta}^2 - \omega^2\sin^2\theta - 2r_0\omega\cos\theta = const} \qquad [11.34]$$

11.13.2. *Painlevé's first integral*

The energy integral [9.11] is not valid here, as the position P of a current particle in the system depends explicitly on the time ($\frac{\partial_{R_g}\overrightarrow{OP}}{\partial t} \neq \vec{0}$). Further, using Newtonian mechanics, we know that the energy integral also does not exist as the engine provides energy to the system.

On the contrary, we can verify that the hypotheses for Painlevé's first integral [9.6] are satisfied, especially because the joints are perfect and the Lagrangian $E^c(q, \dot{q}, t) - \mathcal{V}(q, t)$ is *time independent* (here, $\mathcal{V} = 0$ and E^c is given by [11.33]). Painlevé's first integral $E^{c(2)} - E^{c(0)} + \mathcal{V} = const$ thus gives

$$\frac{1}{2}ma^2\left[4\dot{\theta}^2 + 4\dot{\varphi}^2 - (3 + 4\sin^2\theta)\omega^2 - 4\omega^2\cos^2\theta\right] = const$$

or, after simplification:

$$\dot{\theta}^2 + \dot{\varphi}^2 = const$$

Further, the equation $\mathscr{L}_\varphi$ gives $\dot{\varphi}^2 = (r_0 - \omega\cos\theta)^2$. Hence

$$\dot{\theta}^2 + (r_0 - \omega\cos\theta)^2 = const$$

This is [11.34], which was found earlier.

11.13.3. *Total parameterization*

To determine the engine torque, we choose the following parameterization, where the constraint equation $\dot{\psi} = \omega$ is classified as complementary:

PARAMETERIZATION.

- Primitive parameters: ψ, θ, φ.
- No primitive constraint.
- Retained parameters: ψ, θ, φ.
- Complementary constraint equation: $\dot{\psi} = \omega$.

The kinetic energy of the system is already calculated in [11.22]. The potential of the weight is zero.

As all joints on $\mathcal{S}$ are perfect, we apply Lagrange's equations [8.6] with the Lagrange multiplier denoted here by λ :

$$\begin{aligned}
\mathscr{L}_\varphi : \quad & \frac{\partial E^c}{\partial \dot{\varphi}} = const \quad \rightarrow \quad \boxed{\dot{\varphi} + \omega \cos\theta = r_0} \\
\mathscr{L}_\theta : \quad & 4\ddot{\theta} - 4\dot{\psi}^2 \sin\theta \cos\theta^2 + 4\left(\dot{\varphi} + \dot{\psi}\cos\theta\right)\dot{\psi}\sin\theta = 0 \\
& \rightarrow \quad \boxed{\ddot{\theta} - \sin\theta\cos\theta\omega^2 + r_0\omega\sin\theta = 0} \\
\mathscr{L}_\psi : \quad & ma^2 \frac{d}{dt}\left[\left(3 + 4\sin^2\theta\right)\dot{\psi} + 4\left(\dot{\varphi} + \dot{\psi}\cos\theta\right)\cos\theta\right] = \lambda \\
& \rightarrow \quad \boxed{\lambda = ma^2 \frac{d}{dt}\left[\left(3 + 4\sin^2\theta\right)\omega + 4r_0\cos\theta\right]}
\end{aligned}$$

Let us show that the Lagrange multiplier λ is equal to the desired engine torque. By applying relationship [8.10] to the complementary constraint equation $\dot{\psi} = \omega$, we know that the virtual power of the constraint efforts $\mathcal{F}_{\text{engine}\rightarrow S_1}$ is

$$\mathscr{P}^*(\mathcal{F}_{\text{engine}\rightarrow S_1}) = \lambda\dot{\psi} \qquad [11.35]$$

On the other hand, the virtual power of the engine torque $\vec{\Gamma} = \Gamma\vec{z}_g$ is

$$\begin{aligned}
\mathscr{P}^*(\mathcal{F}_{\text{engine}\rightarrow S_1}) &= \vec{\Gamma}\cdot\vec{\Omega}^*_{g1} \quad \text{where } \vec{\Omega}_{g1} = \dot{\psi}\vec{z}_g \underset{[4.22]}{\Rightarrow} \vec{\Omega}^*_{g1} = \dot{\psi}^*\vec{z}_g \\
&= \Gamma\cdot\dot{\psi}^*
\end{aligned} \qquad [11.36]$$

By comparing [11.35] and [11.36], we find $\lambda = \Gamma$.

REMARK. The meaning of λ may also be found by means of Newtonian mechanics. The theorem of the dynamic moment about axis $O\vec{z}_g$ gives (δ and σ denote the dynamic moment and the angular momentum, respectively):

$$\Gamma = \vec{\delta}_{g\mathscr{S}}(O)\cdot\vec{z}_g = \frac{d}{dt}\sigma_{g\mathscr{S}}(O) = \text{the left-hand side of } \mathscr{L}_\psi = \lambda \qquad \blacksquare$$

We finally obtain the expression for the engine torque that ensures the constraint $\dot{\psi} = \omega$:

$$\boxed{\Gamma = \lambda = ma^2 \frac{d}{dt}\left[\left(3 + 4\sin^2\theta\right)\omega + 4r_0\cos\theta\right] = 4ma^2\dot{\theta}\sin\theta\left(2\omega\cos\theta - r_0\right)}$$

11.14. Rigid body with a non-perfect joint

A rigid body (S) is made up of a homogeneous disk (of center C, radius a and mass m) and a rod AC (with length $a\sqrt{3}$ and negligible mass), which is welded perpendicularly to the disk.

The rigid body (S) is moving in the Galilean reference frame $R_g = R_0$ equipped with a right-handed orthonormal coordinate system $(O; \vec{x}_g, \vec{y}_g, \vec{z}_g)$ in the following manner:

- The end A of the rod coincides with the origin O. The joint at this point is a perfect ball joint between (S) and the support fixed with respect to R_g.
- (S) is in contact at a point I with a horizontal disk (D), the contact takes place with the coefficient of friction f. The disk (D) is connected to the support by a perfect pivot, whose axis is $O\vec{z}_g$, and is rotated by an engine at a constant angular velocity $\omega\vec{z}_g$, $\omega > 0$.

The gravitational acceleration is $-g\vec{z}_g$.

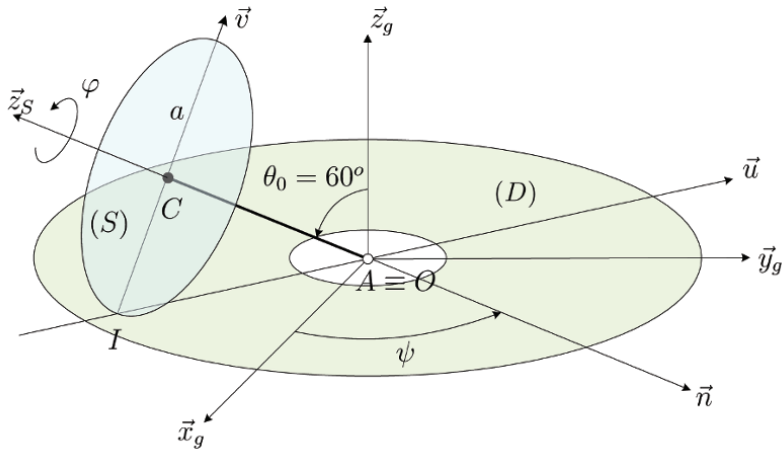


Figure 11.18. *Rigid body with a non-perfect joint*

11.14.1. *Equations set*

We wish to calculate the contact force exerted by (D) on (S), without calculating the constraint force due to the ball joint at O. In order to do this, we choose the following parameterization:

PARAMETERIZATION.

- Primitive parameters: The three Cartesian coordinates x_A, y_A, z_A of the point A, and the three usual Euler angles ψ, θ, φ.
- Primitive constraint equations: $x_A = y_A = z_A = 0$.
- Retained parameters: ψ, θ, φ.
- Complementary constraint equation: $\theta = 60°$.

Equation [8.6] with Lagrange multipliers is not applicable here as the joint at I is not perfect during the slipping stage. We must use the general Lagrange equations [6.5], where

1. the only given effort is the weight, which is derivable from a potential $\mathcal{V}$,
2. the constraint efforts giving rise to the generalized force L_i are the efforts $\mathcal{F}_{\text{ball joint}\to S}$ due to the ball joint exerted on (S) and the efforts $\mathcal{F}_{D\to S}$ exerted by the disk (D) on (S).

The potential of the weight is

$$\mathcal{V} = mga\sqrt{3}\cos\theta + const$$

The virtual power of the efforts due to the ball joint at O is

$$\mathscr{P}^*(\mathcal{F}_{\text{ball joint}\to S}) = \vec{R}_{\text{ball joint}\to S} \cdot \vec{V}^*_{gS}(A) = 0 \qquad \text{since } \vec{V}^*_{gS}(A) = \vec{0}$$

This result is predictable as the ball joint is perfect and the VVF associated with the chosen parameterization is compatible with this joint (refer once again to definition [7.2]).

The virtual power of the contact efforts at I is

$$\mathscr{P}^*(\mathcal{F}_{D\to S}) = \vec{R}_{D\to S}\cdot\vec{V}^*_{gS}(I)$$

We resolve the contact force $\vec{R}_{D\to S}$ exerted by (D) on (S) as $\vec{R}_{D\to S} = N\vec{z}_g + T\vec{n}$, with N and T thus denoting the normal force and the tangential force, respectively. Strictly speaking, the problem is hyperstatic in general and we must also count another tangential force parallel to $\vec{u}$. However, this force is actually not involved in the calculations and may be ignored.

The VV $\vec{V}^*_{gS}(I)$ is calculated in the second Euler intermediary basis $v \equiv (\vec{n}, \vec{v}, \vec{z}_S)$ (refer again to notations [2.3], here $\vec{z}_S$ is the unit vector orienting AC):

$$\begin{aligned}\vec{V}^*_{gS}(I) &= \vec{V}^*_{gS}(A) + \vec{\Omega}^*_{gS}\times\overrightarrow{AI}\\ &= \vec{0} + \left|\begin{matrix}\dot{\theta}^*\\ \dot{\psi}^*\sin\theta\\ \dot{\varphi}^* + \dot{\psi}^*\cos\theta\end{matrix}\right._v \times \left|\begin{matrix}0\\ -a\\ a\sqrt{3}\end{matrix}\right._v = a\left|\begin{matrix}\sqrt{3}\dot{\psi}^*\sin\theta + \dot{\varphi}^* + \dot{\psi}^*\cos\theta\\ -\sqrt{3}\dot{\theta}^*\\ -\dot{\theta}^*\end{matrix}\right._v\end{aligned}$$

Hence

$$\mathscr{P}^*(\mathcal{F}_{D\to S}) = -2aN\dot{\theta}^* + aT(2\dot{\psi}^* + \dot{\varphi}^*)$$

Knowing that the inertia matrix about A of (S) in the basis v is

$$\mathbb{I}_S(A;v) = ma^2\begin{pmatrix}13/4 & 0 & 0\\ 0 & 13/4 & 0\\ 0 & 0 & 1/2\end{pmatrix}$$

we obtain the kinetic energy of (S):

$$E^c_{gS} = \frac{1}{2}\vec{\Omega}_{gS}.\mathcal{I}_S(A)\vec{\Omega}_{gS} = \frac{1}{2}ma^2\left[\frac{13}{4}(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta) + \frac{1}{2}(\dot{\varphi} + \dot{\psi}\cos\theta)^2\right]$$

Finally, Lagrange's equations [6.5] give

$$\mathscr{L}_\psi : \qquad \boxed{ma(41\ddot{\psi} + 4\ddot{\varphi}) = 32T} \qquad [11.37]$$

$$\mathscr{L}_\theta : \qquad \boxed{-\frac{13\sqrt{3}}{4}ma\dot{\psi}^2 + \frac{\sqrt{3}}{2}ma(2\dot{\varphi} + \dot{\psi})\dot{\psi} = 6mg - 8N} \qquad [11.38]$$

$$\mathscr{L}_\varphi : \qquad \boxed{\frac{1}{4}ma(\ddot{\psi} + 2\ddot{\varphi}) = T} \qquad [11.39]$$

This system provides three equations for four unknowns (ψ, φ, T, N). We will write the lacking fourth equation using the Coulomb friction law at the point I. Let us assume that at the initial instant $t = 0$, the rigid body (S) is at rest relative to R_g. Thus, there is slipping between (S) and (D) at the initial instant and, by continuity, *slip prevails at* I, at least in an early stage of the motion.

The slip velocity at I is calculated in the real configuration and we can thus make $\theta = 60°$:

$$\vec{V}_{DS}(I) = \vec{V}_{DS}(A) + \vec{\Omega}_{DS}\times\overrightarrow{AI} = \vec{0} + (\dot{\psi}\vec{z}_g + \dot{\varphi}\vec{z}_S - a\omega\vec{z}_g)\times(-a\vec{v} + a\sqrt{3}\vec{z}_S) = a(2\dot{\psi} + \dot{\varphi} - 2\omega)\vec{n}$$

The slip velocity vector is parallel to $\vec{n}$. Its algebraic measure along $\vec{n}$ is denoted by U:

$$\boxed{U \equiv \vec{V}_{DS}(I)\cdot\vec{n} = v - 2a\omega} \qquad \text{where } \boxed{v \equiv a(2\dot{\psi} + \dot{\varphi})}$$

In the first stage of the motion, where there is slip at I, the Coulomb laws of contact give both $|T| = fN$ and $T\vec{n}.\vec{V}_{DS}(I) = TU < 0$. Further, still in the first stage of the motion, U has the same sign as $U_0 = -2a\omega$, that is $U < 0$ and, thus $T > 0$. The Coulomb laws thus give

$$\boxed{T = fN} \qquad [11.40]$$

Systems [11.37–11.40] form four equations for four unknowns (ψ, φ, T, N).

11.14.2. *Solving the equations set*

By eliminating T from [11.37] and [11.39], we can express $\dot{\psi}$ and $\dot{\varphi}$ as a function of the auxiliary variable v:

$$\dot{\varphi} = \frac{11}{19}\frac{v}{a} \quad \dot{\psi} = \frac{4}{19}\frac{v}{a} \qquad [11.41]$$

Equations [11.37] (or [11.39]) and [11.40] thus give

$$T = \frac{13}{38}m\dot{v} \quad N = \frac{13}{38}\frac{m\dot{v}}{f} \qquad [11.42]$$

Inserting [11.41] and [11.42] in [11.38] yields

$$\dot{v} = \frac{57}{26}gf = const > 0 \qquad \Rightarrow \qquad v = \frac{57}{26}gft$$

From this, we finally obtain

$$\boxed{\psi = \frac{3}{13}\frac{gf}{a}t^2 + \psi_0} \quad \boxed{\varphi = \frac{33}{52}\frac{gf}{a}t^2 + \varphi_0} \quad \boxed{T = \frac{3}{4}mgf > 0} \quad \boxed{N = \frac{3}{4}mg > 0}$$

The slip ends when U vanishes, that is, at the instant $t_1 = \frac{52}{57}\frac{a\omega}{gf}$. We will not study the second stage of the motion, which takes place after t_1 and we admit that there is no slip for $t \geq t_1$.

11.14.3. *Power of the engine and work dissipated through friction*

We will calculate the work done by the engine that causes the disk (D) to rotate and we will examine how this work is split into the energy dissipated through friction at the point I and the increase in the kinetic energy of (S).

The kinetic energy theorem applied to the disk (D) alone gives, during the slipping stage:

$$\mathscr{P}(\mathcal{F}_{\text{int}\to D}) + \mathscr{P}(\mathcal{F}_{\text{ext}\to D}) = \frac{d}{dt}E^c_{gD} \qquad [11.43]$$

where the power $\mathscr{P}(\mathcal{F}_{\text{int}\to D})$ of the efforts internal to (D) is zero, as (D) is a rigid body and the power of the external efforts can be divided into three components: the power of the weight of (D), the power of the efforts of S on D and the power of the engine:

$$\mathscr{P}(\mathcal{F}_{\text{ext}\to D}) = \mathscr{P}(\mathcal{F}_{\text{weight}\to D}) + \mathscr{P}(\mathcal{F}_{S\to D}) + \mathscr{P}(\mathcal{F}_{\text{engine}\to D})$$

We have

$$\mathscr{P}(\mathcal{F}_{\text{weight}\to D}) = 0 \qquad \text{since the center of mass of } (D) \text{ is fixed in } R_g$$
$$\mathscr{P}(\mathcal{F}_{S\to D}) = -(N\vec{z}_g + T\vec{n}) \cdot \vec{V}_{gD}(I) = -2aT\omega = -\frac{13}{38}ma\omega\dot{v}$$

From [11.43], we derive the power provided by the engine

$$\mathscr{P}(\mathcal{F}_{\text{engine}\to D}) = \frac{13}{38}ma\omega\dot{v}$$

The work done by the engine during the slipping stage, denoted by W, is obtained by integrating the power between the instants 0 and t_1:

$$\boxed{W \equiv W(\mathcal{F}_{\text{engine}\to D}, [0, t_1]) = \frac{26}{19} m a^2 \omega^2}$$

Let us now apply the theorem of kinetic energy to the whole system $(S) \cup (D)$:

$$\mathscr{P}(\mathcal{F}_{\text{engine}\to D}) + \mathscr{P}(\mathcal{F}_{D\leftrightarrow S}) = \frac{d}{dt}\left(E^c_{gS} + E^c_{gD}\right),$$

where $\mathcal{F}_{D\leftrightarrow S}$ denotes the inter-efforts between (D) and (S), i.e. the inter-forces of contact at I. Upon integration over time between 0 and t_1, this gives:

$$W + \int_0^{t_1} \mathscr{P}(\mathcal{F}_{D\leftrightarrow S})dt = E^c_{gS}(t_1) - \underbrace{E^c_{gS}(0)}_{=0} + \underbrace{E^c_{gD}(t_1) - E^c_{gD}(0)}_{=0} \qquad [11.44]$$

Further, the work of the inter-efforts of contact at I over the time interval $[0, t_1]$ is given as

$$\begin{aligned}\int_0^{t_1} \mathscr{P}(\mathcal{F}_{D\leftrightarrow S})dt &= \int_0^{t_1} \vec{R}_{D\to S} \cdot \vec{V}_{DS}(I)dt = \int_0^{t_1} (N\vec{z}_g + T\vec{n}) \cdot (v - 2a\omega)\,\vec{n}dt \\ &= -\frac{13}{19} m a^2 \omega^2 = -\frac{W}{2}\end{aligned}$$

We thus finally obtain $\boxed{E^c_{gS}(t_1) = \frac{W}{2}}$.

By rewriting equality [11.44] in the form

$$\underbrace{W + \int_0^{t_1} \mathscr{P}(\mathcal{F}_{D\leftrightarrow S})dt}_{-W/2} = \underbrace{E^c_{gS}(t_1)}_{W/2}$$

it can be seen that during the slipping stage:

– half the work done by the engine is dissipated through friction,

– while the other half is used to give (S) enough kinetic energy for the slip at I to stop.

Appendix 1

Tensors

Tensor theory is used in several fields of physics, especially in mechanics where it is absolutely essential for the study of deformable media (deformable solids or fluids). A comprehensive presentation on tensors would be very long and, in fact, unnecessary within the scope of this book. Indeed, the mechanics of rigid bodies only requires what is called "second-order tensors" in the general tensor theory. Since these can be identified with linear mappings, the corresponding results are rather well known.

This appendix brings together the basic results related to second-order tensors, which will simply be referred to as "tensors". These results, which are derived from the general tensor theory, are sufficient for the purpose of this book.

Consider a three-dimensional Euclidian space. The scalar product of two vectors $\vec{x}, \vec{y} \in E$ is denoted by $\vec{x}.\vec{y}$.

Definition. In this book, *a tensor* is a linear mapping from E to E. The two terms "tensor" and "linear mapping" are, thus, *synonymous*. [A1.1]

A tensor is usually denoted by a letter with two bars above it, similar to a vector which is a letter with an arrow over it.

Consider a tensor $\bar{\bar{T}}$, its image $\bar{\bar{T}}(\vec{x})$ of a vector $\vec{x}$ of E is a vector of E. As is common for linear mappings, the image of a vector x under a linear mapping f is denoted by $f.x$ instead of $f(x)$, using the same "dot" symbol as used for the scalar product. We will thus write $\bar{\bar{T}}.\vec{x}$ instead of $\bar{\bar{T}}(\vec{x})$:

$$\begin{array}{rcl} \bar{\bar{T}} : E & \rightarrow & E \\ \vec{x} & \mapsto & \bar{\bar{T}}.\vec{x} \end{array} \qquad \text{[A1.2]}$$

The following tensors are encountered in the mechanics of rigid bodies:

1. In section 1.3.5, the tensor of the type $\bar{\bar{Q}}_{21}$, which is *the rotation tensor of a reference frame R_1 with respect to another reference frame R_2*, or *the reference frame change tensor*.

2. In section 1.6, the tensor of the type $\bar{\bar{\Omega}}_{12}$, which is called *the angular velocity tensor of R_2 with respect to R_1*.

Representative matrix of a tensor

As with any linear mapping, one speaks of *the representative matrix of a tensor in a given basis*. Let $e = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be a basis of E, the representative matrix of a tensor $\bar{\bar{T}}$ in e is the following 3×3 matrix:

$$\boxed{Mat(\bar{\bar{T}}; e) = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}} \text{ where } \forall i, j \in \{1, 2, 3\}, \boxed{T_{ij} = \vec{e}_i.(\bar{\bar{T}}.\vec{e}_j)} \qquad \text{[A1.3]}$$

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A tensor is completely determined when its representative matrix in the basis e is determined and vice versa.

Transpose of a tensor

Definition. The *transpose* of a tensor $\bar{\bar{T}}$, denoted $\boxed{\bar{\bar{T}}^T}$, is the tensor that satisfies

$$\forall \vec{x}, \vec{y} \in E, \boxed{\vec{x}.(\bar{\bar{T}}.\vec{y}) = \vec{y}.(\bar{\bar{T}}^T.\vec{x})} \qquad \text{[A1.4]}$$

The representative matrix of the transpose tensor $\bar{\bar{T}}^T$ in basis e is the transpose of the matrix of $\bar{\bar{T}}$ in the same basis, which justifies the terminology.

Symmetric tensor: skew-symmetric tensor

Definition.

1. A tensor $\bar{\bar{T}}$ is *symmetric* if it is equal to its transpose: $\bar{\bar{T}} = \bar{\bar{T}}^T$.
2. A tensor $\bar{\bar{T}}$ is *skew-symmetric* if it is equal to the opposite of its transpose: $\bar{\bar{T}}^T = -\bar{\bar{T}}$.

The representative matrix of a symmetric (respectively, skew-symmetric) tensor is symmetric (respectively, skew-symmetric).

According to the previous definition and relationship [A1.4], we have the following results:

$$\begin{array}{lll} \bar{\bar{T}} \text{ is symmetric} & \Leftrightarrow & \forall \vec{x}, \vec{y} \in E, \vec{x}.(\bar{\bar{T}}.\vec{y}) = \vec{y}.(\bar{\bar{T}}.\vec{x}) \\ \bar{\bar{T}} \text{ is skew-symmetric} & \Leftrightarrow & \forall \vec{x}, \vec{y} \in E, \vec{x}.(\bar{\bar{T}}.\vec{y}) = -\vec{y}.(\bar{\bar{T}}.\vec{x}) \end{array} \qquad \text{[A1.5]}$$

Identity tensor

Definition. *The identity tensor*, denoted by $\boxed{\bar{\bar{I}}}$, is the tensor whose image of any vector $\vec{x}$ is the vector $\vec{x}$ itself:

$$\forall \vec{x} \in E, \boxed{\bar{\bar{I}}.\vec{x} = \vec{x}} \qquad \text{[A1.6]}$$

The representative matrix of the identity tensor in the basis e is the unit matrix of the third order.

Product of two tensors

Definition. The *product* of a tensor $\bar{\bar{S}}$ and another tensor $\bar{\bar{T}}$, denoted by $\boxed{\bar{\bar{S}}.\bar{\bar{T}}}$, is, by definition, the composite of the linear mappings $\bar{\bar{S}}$ and $\bar{\bar{T}}$. The usual symbol $\circ$ for the composition of functions is, here, replaced by the dot.

The representative matrix of the product $\bar{\bar{S}}.\bar{\bar{T}}$ in the basis e is the product of the representative matrices of $\bar{\bar{S}}$ and $\bar{\bar{T}}$ in the same basis, which justifies the term "product" and the "dot" used for the product of tensors.

In fact, the use of the "dot" symbol has a deeper origin in tensor algebra. It represents the so-called *singly contracted product* of two tensors. The scalar product $\vec{x}.\vec{y}$ of two vectors and the image $\bar{\bar{T}}.\vec{u}$ of a vector under a tensor are particular cases of the singly contracted product of two vectors. This explains why the same "dot" symbol is used in all these operations.

Inverse tensor

We can easily verify the following theorem:

Theorem and Definition. Let $\bar{\bar{T}}$ be a tensor. If there exists a tensor $\bar{\bar{S}}$ satisfying $\bar{\bar{S}}.\bar{\bar{T}} = \bar{\bar{T}}.\bar{\bar{S}} = \bar{\bar{I}}$, then such a tensor $\bar{\bar{S}}$ is unique. It is called the *inverse* of tensor $\bar{\bar{T}}$ and is denoted by $\boxed{\bar{\bar{T}}^{-1}}$.

The inverse of the tensor $\bar{\bar{T}}$ is, thus, the inverse linear mapping of $\bar{\bar{T}}$.

The representative matrix of the inverse tensor $\bar{\bar{T}}^{-1}$ in the basis e is the inverse of the matrix of $\bar{\bar{T}}$ in the same basis.

Orthogonal tensor

Definition. A tensor $\bar{\bar{T}}$ is *orthogonal* if its inverse is equal to its transpose: $\bar{\bar{T}}^{-1} = \bar{\bar{T}}^{T}$.

An orthogonal tensor is a vector isometry and represents a rotation in mechanics.

Tensor product

Definition. *The tensor product* of two vectors $\vec{a}$ and $\vec{b}$, denoted by $\boxed{\vec{a} \otimes \vec{b}}$, is the tensor defined as

$$\forall \vec{c} \in E, \boxed{(\vec{a} \otimes \vec{b}).\vec{c} = \vec{a}(\vec{b}.\vec{c})} \qquad \text{[A1.7]}$$

The representative matrix of the tensor product $\vec{a} \otimes \vec{b}$ is easy to obtain. Its (i, j) component is given by

$$\forall i, j \in \{1, 2, 3\}, \boxed{(\vec{a} \otimes \vec{b})_{ij} = a_i b_j} \qquad \text{[A1.8]}$$

We can easily verify the following result:

Theorem.

$$\forall \vec{a}, \vec{b} \in E, \boxed{\left(\vec{a} \otimes \vec{b}\right)^{T} = \vec{b} \otimes \vec{a}} \qquad \text{[A1.9]}$$

Appendix 2

Typical Perfect Joints

In this appendix, we consider two rigid bodies S_1 and S_2, connected by one of the following joints, which are usually encountered in mechanics:

1. Point contact.
2. Ball-and-socket joint (or spherical joint).
3. Cylindrical joint.
4. Pivot or hinged joint
5. Prismatic or sliding joint.
6. Helical joint (or screw joint).

It is assumed that the joint under study is *perfect* in the sense of definition [7.79]: the joint is perfect if, at any instant t, the VP of the constraint inter-efforts between S_1 and S_2 is zero in *any VVF* V^* *compatible* with this joint:

$$\forall t, \forall \text{ VVF } V^* \textit{ compatible} \text{ with this joint}, \mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = 0 \qquad \text{[A2.1]}$$

(recall that the VP of inter-efforts is written without the reference frame index as it is independent of the reference frame).

We will show that for the joint to be perfect, it is necessary and sufficient for the constraint efforts to satisfy a certain number of specific conditions.

We will adopt the following general notations:

– $(O; b_0) \equiv (O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$: orthonormal coordinate system fixed in the common reference frame R_0;

– O_1: point attached to the rigid body S_1; $b_1 \equiv (\vec{x}_1, \vec{y}_1, \vec{z}_1)$: orthonormal basis related to S_1;

  $(\psi_1, \theta_1, \varphi_1)$: the Euler angles defining the position of basis b_1 with respect to basis b_0;

  R_1: the reference frame defined by S_1,

– O_2: point attached to the rigid body S_2; $b_2 \equiv (O_2; \vec{x}_2, \vec{y}_2, \vec{z}_2)$: orthonormal basis attached to S_2;

  $(\psi_2, \theta_2, \varphi_2)$: the Euler angles defining the position of basis b_2 with respect to basis b_0.

Other notations will be introduced depending on the joint being studied.

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A2.1. Point contact between two rigid bodies

Consider two rigid bodies S_1, S_2, which remain in contact, throughout their motion, at a point I that may or may not be variable. The two contact surfaces are assumed to be regular, such that there exists at I a tangent plane Π, common to both contact surfaces (Figure A2.1).

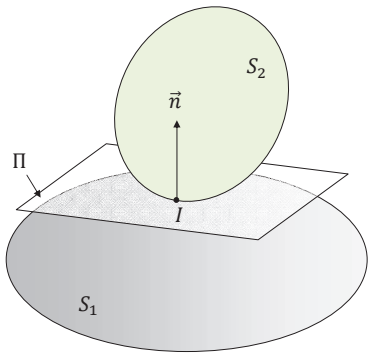

Figure A2.1. *Point contact*

The position of the system consisting of two rigid bodies is defined, *a priori*, by 12 primitive parameters; for example, for each rigid body, we can take three Cartesian coordinates of a particle of the rigid body and three Euler angles. There is actually no use in naming these parameters as they do not come into play in the calculations that follow.

The contact condition between the two rigid bodies is obtained using geometric calculations. Its expression is complicated in the general case when the surfaces have arbitrary forms. Fortunately, in the analysis here, it is sufficient to use the simpler relationship given below, which is a consequence of the contact relationships:

$$\vec{V}_{12}(I).\vec{n} = 0$$

This relationship means that the two rigid bodies do not interpenetrate: the relative velocity at the point I has no component along the normal direction $\vec{n}$ at the same point and must, thus, belong to the tangent plane Π. It is expressed in three dimensions by two scalar, non-holonomic equations, which must thus be counted as complementary constraint equations.

The following result gives *one characterization* for a perfect point contact:

Theorem. The point contact is *perfect* if and only if

$$\boxed{\vec{R}_{1\to 2} \mathbin{/\!/} \vec{n}} \quad \text{and} \quad \boxed{\vec{\mathcal{M}}_{1\to 2}(I) = \vec{0}} \qquad \text{[A2.2]}$$

where $\vec{R}_{1\to 2}$ is the resultant force and $\vec{\mathcal{M}}_{1\to 2}(I)$ is the resultant moment about point I of the contact efforts exerted by S_1 on S_2.

In other words, the point contact is *perfect* if and only if the contact takes place *without friction and without rolling and pivoting resistance*, or again, if the contact efforts at the point I exerted by S_1 on S_2 reduce to a single normal force.

PROOF. The VP of the constraint inter-efforts in [A2.1] is given by [5.14]:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{R}_{1\to 2}.\vec{V}^*_{12}(I) + \vec{\mathcal{M}}_{1\to 2}(I).\vec{\Omega}^*_{12} \qquad \text{[A2.3]}$$

From theorem [7.78], it is known that the VVFs compatible with a mechanical joint are *independent* of the choice of parameterization. We will apply definition [A2.1] by choosing the following parameterization:

PARAMETERIZATION.

- Primitive parameters: Six primitive parameters for each rigid body S_1, S_2; there is no need to explicitly list them here.
- No primitive constraint equation.
- Retained parameters: The same as the primitive parameters.
- Complementary constraint equation: $\vec{V}_{12}(I).\vec{n} = 0$.

Thus, the virtual velocities compatible with the joint satisfy:

$$\vec{V}^*_{12}(I).\vec{n} = 0 \qquad \text{[A2.4]}$$

Let us resolve the contact force $\vec{R}_{1\to 2}$ into a normal component $N_{1\to 2}$ directed along $\vec{n}$ and a tangential component $\vec{T}_{1\to 2}$ lying in plane Π:

$$\vec{R}_{1\to 2} = N_{1\to 2}\vec{n} + \vec{T}_{1\to 2}$$

Taking into account [A2.4], the VP [A2.3] then becomes:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{T}_{1\to 2}.\vec{V}^*_{12}(I) + \vec{\mathcal{M}}_{1\to 2}(I).\vec{\Omega}^*_{12}$$

This power is zero for any VV $\vec{V}^*_{12}(I)$ (belonging to plane Π) and for any virtual angular velocity $\vec{\Omega}^*_{12}$, if and only if $\vec{T}_{1\to 2} = \vec{0}$ and $\vec{\mathcal{M}}_{1\to 2}(I) = \vec{0}$. ■

• If we reinforce the contact condition by assuming that the relative slip velocity $\vec{V}_{12}(I)$ is constantly zero, we have a simpler result:

Theorem. If the contact takes place *without slipping*; i.e. $\vec{V}_{12}(I) = \vec{0}$, then the point contact is *perfect* if and only if

$$\boxed{\vec{\mathcal{M}}_{1\to 2}(I) = \vec{0}} \qquad \text{[A2.5]}$$

PROOF. This proof is similar to the previous one. We choose the same parameterization as above, with the difference that this time the complementary constraint equation is $\vec{V}_{12}(I) = \vec{0}$. The virtual velocities compatible with the joint thus satisfy

$$\vec{V}^*_{12}(I) = \vec{0}$$

The VP [A2.3] then becomes

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{\mathcal{M}}_{1\to 2}(I).\vec{\Omega}^*_{12}$$

This power is zero for any virtual angular velocity $\vec{\Omega}^*_{12}$ if and only if $\vec{\mathcal{M}}_{1\to 2}(I) = \vec{0}$. ■

The reasoning used in this section, as in the rest of this appendix, involves the *relative* motion between the two rigid bodies S_1 and S_2, but *not* the motion that S_1 may have with respect to a given reference frame (for example, the common reference frame R_0). Consequently, the results obtained remain valid if there exist additional constraint equations between the primitive parameters of S_1, for example, if S_1 is fixed in R_0.

A2.2. Ball-and-socket joint (or spherical joint)

Consider two rigid bodies S_1 and S_2, the first containing a spherical cavity with center O_1 and the second containing a spherical ball with center O_2 and whose radius is slightly smaller than the radius of the spherical cavity (Figure A2.2(a)). The rigid bodies are connected by a ball-and-socket joint when the spherical ball is lodged inside the spherical cavity (Figure A2.2(b)), such that the centers O_1, O_2 coincide during the motion. The spherical ball is able to freely rotate in the spherical cavity.

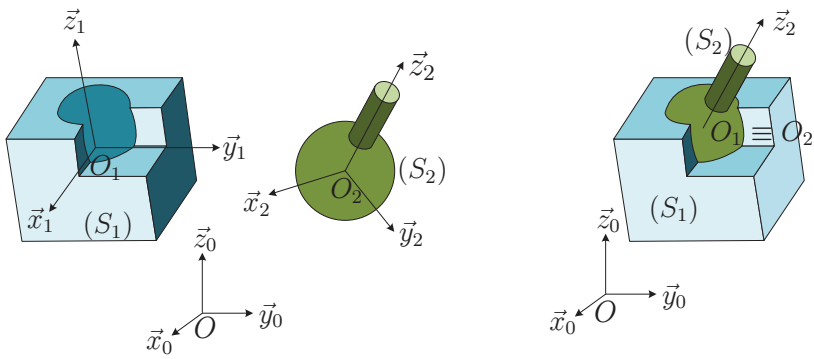

Figure A2.2. *Ball-and-socket joint*

Theorem. The spherical joint is *perfect* if and only if

$$\boxed{\vec{\mathcal{M}}_{1\to 2}(O_1) = \vec{0}}, \qquad [A2.6]$$

where $\vec{\mathcal{M}}_{1\to 2}(O_1)$ is the resultant moment about O_1 of the contact efforts exerted by S_1 on S_2.

PROOF. We still use definition [A2.1] for a perfect joint and since the VVFs compatible with a mechanical joint are *independent* of the choice of parameterization, we will carry out the calculations by choosing the following parameterization:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters:
  - Six parameters for S_1: Three coordinates for the point O_1 relative to the coordinate system $(O; b_0)$ and the three Euler angles $\psi_1, \theta_1, \varphi_1$;
  - Six parameters for S_2: Three coordinates for the point O_2 relative to the coordinate system $(O; b_0)$ and the three Euler angles $\psi_2, \theta_2, \varphi_2$.
- Primitive constraint equation: $O_1 = O_2$, which amounts to saying that the coordinates of points O_1 and O_2 are identical.
- Retained parameters: $\psi_1, \theta_1, \varphi_1$ and $\psi_2, \theta_2, \varphi_2$.
- No complementary constraint equation.

The VP of the constraint inter-efforts is given by [5.14]:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{R}_{1\to 2}.\vec{V}^*_{12}(O_2) + \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{\Omega}^*_{12} \qquad \text{[A2.7]}$$

Let o_2 be the particle of S_2 whose position at any instant is O_2. Then, $\vec{V}^*_{12}(O_2)$ is the VV with respect to R_1 of the particle o_2 (see notation [4.33]). Let us calculate this VV using definition [4.10]:

$$\begin{aligned} \vec{V}^*_{12}(O_2) &= \vec{V}^*_{R_1}(o_2) \equiv \bar{\bar{Q}}_{01}.\sum_{i=1}^{6} \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1O_2}\right)\dot{q}^*_i \\ &= \vec{0} \qquad \text{as } \overrightarrow{O_1O_2} = \vec{0} \end{aligned} \qquad \text{[A2.8]}$$

We may obtain the same result when using the composition formula for virtual velocities [4.49], $\vec{V}^*_{12}(O_2) = \vec{V}^*_{02}(O_2) - \vec{V}^*_{01}(O_2) = \vec{V}^*_{02}(O_2) - \vec{V}^*_{01}(O_1)$, however the calculations are a little longer.

Let us also calculate the virtual angular velocity $\vec{\Omega}^*_{12}$ using the composition formula [4.45], $\vec{\Omega}^*_{12} = \vec{\Omega}^*_{02} - \vec{\Omega}^*_{01}$. The virtual angular velocities are given by [4.23]:

$$\vec{\Omega}^*_{01} = \dot{\psi}^*_1\vec{z}_0 + \dot{\theta}^*_1\vec{n}_1 + \dot{\varphi}^*_1\vec{z}_1 \qquad \vec{\Omega}^*_{02} = \dot{\psi}^*_2\vec{z}_0 + \dot{\theta}^*_2\vec{n}_2 + \dot{\varphi}^*_2\vec{z}_2 \qquad \text{[A2.9]}$$

where $\vec{n}_1$ (respectively, $\vec{n}_2$) is the vector of the line of nodes of S_1 (respectively, S_2), the triplets $(\dot{\psi}^*_1, \dot{\theta}^*_1, \dot{\varphi}^*_1)$ and $(\dot{\psi}^*_2, \dot{\theta}^*_2, \dot{\varphi}^*_2)$ are arbitrary.

The VVF obtained is automatically compatible with the joint being considered, since there is no complementary constraint equation. Taking into account [A2.8] and [A2.9], the VP [A2.7] becomes:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{\Omega}^*_{12}$$

where $\vec{\Omega}^*_{12}$ is an arbitrary vector. This power is zero for any virtual angular velocity $\vec{\Omega}^*_{12}$, if and only if $\vec{\mathcal{M}}_{1\to 2}(O_2) = \vec{0}$. ■

Relationship [A2.6], which characterizes the perfection of the spherical joint, provides *global* information on the constraint efforts *field*. However, it does not give any information on the distribution of these efforts over the contact surface between the two rigid bodies. This observation also applies to the other types of joints studied in this appendix.

A2.3. Cylindrical joint

Consider two rigid bodies S_1 and S_2, the first containing a circular cylindrical cavity with axis $O_1\vec{z}_1$ and the second consisting of a cylinder with axis $O_2\vec{z}_2$ and with a radius slightly smaller than the radius of the cavity (Figure A2.3(a)). The two rigid bodies are connected by a cylindrical joint when the cylinder slides into and rotates within the cavity, such that the two axes coincide during the motion (Figure A2.3(b)).

Theorem. The cylindrical joint is *perfect* if and only if

$$\boxed{\vec{R}_{1\to 2} \perp \vec{z}_1 \qquad \text{and} \qquad \vec{\mathcal{M}}_{1\to 2}(O_2) \perp \vec{z}_1} \qquad \text{[A2.10]}$$

In other words, the cylindrical joint is *perfect* if and only if the resultant force and the resultant moment (about one point on the axis) of the constraint efforts are perpendicular to the cylinder axis.

FIRST PROOF. Let us first write out the constraint equations, independent of whether they are classified as primitive or complementary equations.

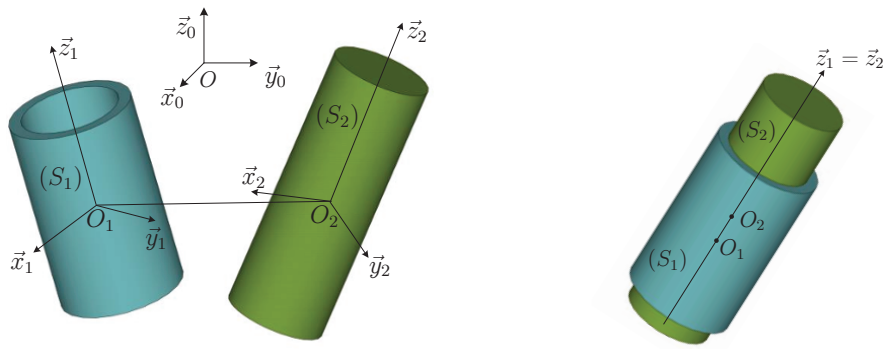

Figure A2.3. *Cylindrical joint*

The primitive parameters for S_1 are the conventional parameters: the Cartesian coordinates (x_1, y_1, z_1) of point O_1 relative to the coordinate system $(O; b_0) \equiv (O; \vec{x}_0, \vec{y}_0, \vec{z}_0)$, and the Euler angles $\psi_1, \theta_1, \varphi_1$.

For S_2, we take the conventional Euler angles $\psi_2, \theta_2, \varphi_2$, but in order to simplify the calculations, we choose the Cartesian coordinates (ξ_2, η_2, ζ_2) of point O_2 relative to the coordinate system $(O_1; b_1) \equiv (O_1; \vec{x}_1, \vec{y}_1, \vec{z}_1)$, instead of those relative to the coordinate system $(O; b_0)$. The coordinates (ξ_2, η_2, ζ_2) are the components of vector $\overrightarrow{O_1O_2}$ in the basis b_1 attached to S_1.

We avoid using the Euler angles $(\alpha_2, \beta_2, \gamma_2)$ defining the position of basis b_2 relative to basis b_1, as the constraint $O_2\vec{z}_2 = O_1\vec{z}_1$ entails a singularity: the nutation $\beta_2 = 0$ and only the sum $\alpha_2 + \gamma_2$ of precession and spin is determined.

The cylindrical joint can be expressed by

$$\text{axis } O_2\vec{z}_2 = \text{ axis } O_1\vec{z}_1 \Leftrightarrow \begin{cases} O_2 \in \text{ axis } O_1\vec{z}_1 & \Leftrightarrow \xi_2 = \eta_2 = 0 \\ \vec{z}_2 = \vec{z}_1 (\Rightarrow \vec{n}_1 = \vec{n}_2) & \Leftrightarrow \psi_2 = \psi_1, \theta_2 = \theta_1 \end{cases} \qquad \text{[A2.11]}$$

that is, by four scalar constraint equations.

We work with the following parameterization:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters:
  - six for S_1 : (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - six for S_2 : (ξ_2, η_2, ζ_2) and $(\psi_2, \theta_2, \varphi_2)$.
- Primitive constraint equations: four equations [A2.11].
- Retained parameters:
  - six for S_1 : (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - two for S_2: the coordinate ζ_2 of point O_2 along the axis $O_1\vec{z}_1$ and the spin angle φ_2.
- No complementary constraint equation.

The VP of the constraint inter-efforts is given by [5.14]:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{R}_{1\to 2}.\vec{V}^*_{12}(O_2) + \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{\Omega}^*_{12} \qquad \text{[A2.12]}$$

Let us calculate the VV $\vec{V}^*_{12}(O_2)$ using definition [4.10], with o_2 denoting the particle of S_2 located at O_2 at any instant, as in the analysis of spherical joint:

$$\vec{V}^*_{12}(O_2) = \vec{V}^*_{R_1}(o_2) \equiv \bar{\bar{Q}}_{01}.\sum_i \frac{\partial}{\partial q_i}\left(\bar{\bar{Q}}_{10}.\overrightarrow{O_1O_2}\right)\dot{q}^*_i$$

where $\bar{\bar{Q}}_{10}.\overrightarrow{O_1O_2} = \zeta_2\vec{e}_3$ (refer again to [1.30]: $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ is the canonical basis of $\mathbb{R}^3$ and $(\vec{x}_0, \vec{y}_0, \vec{z}_0) = (\vec{e}_1, \vec{e}_2, \vec{e}_3)$). Hence

$$\vec{V}^*_{12}(O_2) = \bar{\bar{Q}}_{01}.\frac{\partial}{\partial \zeta_2}(\zeta_2\vec{e}_3)\,\dot{\zeta}^*_2 = \bar{\bar{Q}}_{01}.\vec{e}_3\,\dot{\zeta}^*_2 = \dot{\zeta}^*_2\vec{z}_1 \qquad \text{[A2.13]}$$

If we calculated the VV $\vec{V}^*_{12}(O_2)$ via the real velocity as indicated in section 4.12:

$$\vec{V}_{12}(O_2) = \frac{d_{R_1}\overrightarrow{O_1O_2}}{dt} = \frac{d_{R_1}}{dt}(\zeta_2\vec{z}_1) = \dot{\zeta}_2\vec{z}_1$$

we would have arrived at the same result.

On the other hand, let us calculate the virtual angular velocity $\vec{\Omega}^*_{12}$ via the (real) relative angular velocity $\vec{\Omega}_{12}$, since the latter is easy to obtain:

$$\vec{\Omega}_{12} = \vec{\Omega}_{02} - \vec{\Omega}_{01} = (\dot{\varphi}_2 - \dot{\varphi}_1)\vec{z}_1$$

From this, we can derive the virtual angular velocity $\vec{\Omega}^*_{12}$ following the procedure described in [4.22]:

$$\vec{\Omega}^*_{12} = (\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\vec{z}_1 \qquad \text{[A2.14]}$$

Taking into account [A2.13] and [A2.14], the VP [A2.12] becomes:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \dot{\zeta}^*_2\,\vec{R}_{1\to 2}.\vec{z}_1 + (\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\,\vec{\mathfrak{M}}_{1\to 2}(O_2).\vec{z}_1$$

where the scalars $\dot{\zeta}^*_2, \dot{\varphi}^*_1, \dot{\varphi}^*_2$ are arbitrary. ■

SECOND PROOF. Below is given another an intrinsic proof, which does not use the coordinate ζ_2. Let us write the constraint equation $O_2 \in$ axis $O_1\vec{z}_1$ as $\overrightarrow{O_1O_2} \times \vec{z}_1 = \vec{0}$ and derive this relationship with respect to time relative to the reference frame R_1:

$$\frac{d_{R_1}\overrightarrow{O_1O_2}}{dt} \times \vec{z}_1 + \overrightarrow{O_1O_2} \times \underbrace{\frac{d_{R_1}\vec{z}_1}{dt}}_{\vec{0}} = \vec{0} \qquad \Leftrightarrow \qquad \vec{V}_{12}(O_2) \times \vec{z}_1 = \vec{0}$$

According to the procedure described in section 4.12, it follows that the VVF $\vec{V}^*_{12}(O_2)$ satisfies a similar relationship:

$$\vec{V}^*_{12}(O_2) \times \vec{z}_1 = \vec{0} \qquad \text{i.e.} \qquad \vec{V}^*_{12}(O_2) = \left(\vec{V}^*_{12}(O_2).\vec{z}_1\right)\vec{z}_1 \qquad \text{[A2.15]}$$

Taking into account [A2.15] and [A2.14], the VP [A2.12] can now be written as

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \left(\vec{V}^*_{12}(O_2).\vec{z}_1\right)\left(\vec{R}_{1\to 2}.\vec{z}_1\right) + (\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\,\vec{\mathfrak{M}}_{1\to 2}(O_2).\vec{z}_1$$

where the scalars $\vec{V}^*_{12}(O_2).\vec{z}_1, \dot{\varphi}^*_1, \dot{\varphi}^*_2$ are arbitrary. ■

REMARK. Let us return to equations [A2.11] expressing the cylindrical joint. To obtain it, we could have chosen, as the primitive parameters for S_2, the coordinates (x_2, y_2, z_2) of point

O_2 relative to the coordinate system $(O; b_0)$, instead of the coordinates (ξ_2, η_2, ζ_2) of point O_2 relative to the coordinate system $(O_1; b_1)$. However, the constraint equations that result from this will be longer. Indeed, since $\vec{z}_1 = -\sin\theta_1 \vec{u}_1 + \cos\theta_1 \vec{z}_0 = -\sin\theta_1(\cos\psi_1 \vec{y}_0 - \sin\psi_1 \vec{x}_0) + \cos\theta_1 \vec{z}_0$, the constraint $O_2 \in$ axis $O_1\vec{z}_1$ is expressed by

$$\overrightarrow{O_1O_2} \times \vec{z}_1 = \vec{0} \Leftrightarrow \begin{vmatrix} x_2 - x_1 \\ y_2 - y_1 \\ z_2 - z_1 \end{vmatrix} \times \begin{vmatrix} \sin\theta_1 \sin\psi_1 \\ -\sin\theta_1 \cos\psi_1 \\ \cos\theta_1 \end{vmatrix} = 0$$

$$\Leftrightarrow \begin{cases} (y_2 - y_1)\cos\theta_1 + (z_2 - z_1)\sin\theta_1 \cos\psi_1 = 0 \\ (z_2 - z_1)\sin\theta_1 \sin\psi_1 - (x_2 - x_1)\cos\theta_1 = 0 \\ (x_2 - x_1)\cos\psi_1 - (y_2 - y_1)\sin\psi_1 = 0 \end{cases} \qquad [A2.16]$$

Only two of the three equations [A2.16] are independent. We can retain, for example, $[A2.16]_1$ and $[A2.16]_2$ to eliminate x_2, y_2 in favor of z_2 and (x_1, y_1, z_1). The expressions obtained are more complicated than $\xi_2 = \eta_2 = 0$ in $[A2.11]_1$. ■

A2.4. Pivot (or hinged joint)

The pivot (or hinged) joint between two rigid bodies S_1 and S_2 is a cylindrical joint where the relative translation of the two rigid bodies along their common axis $O_1\vec{z}_1$ is prevented (Figure A2.4), that is, O_1 and O_2 are forced to be identical.

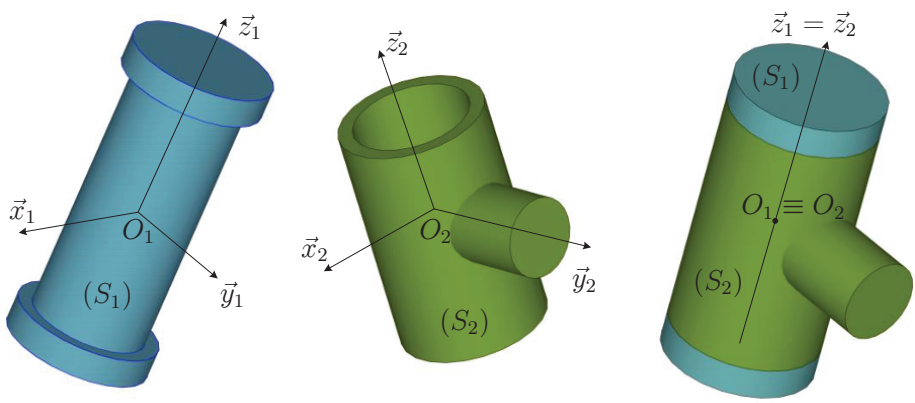

Figure A2.4. *Pivot joint*

The pivot joint is, thus, expressed by relationships [A2.11] to which we add the additional constraint $\zeta_2 = 0$:

$$\begin{cases} O_2 = O_1 & \Leftrightarrow \ \xi_2 = \eta_2 = \zeta_2 = 0 \\ \vec{z}_2 = \vec{z}_1 (\Rightarrow \vec{n}_1 = \vec{n}_2) & \Leftrightarrow \ \psi_2 = \psi_1, \theta_2 = \theta_1 \end{cases} \qquad [A2.17]$$

that is, by five scalar constraint equations. The rigid body S_1 may move freely while the single motion possible for S_2 relative to S_1 is the rotation around the common axis $O_1\vec{z}_1$.

Theorem. The pivot joint is *perfect* if and only if

$$\boxed{\vec{\mathcal{M}}_{1\to 2}(O_2) \perp \overrightarrow{z_1}} \qquad [A2.18]$$

PROOF. We choose the same parameterization as in the analysis of the cylindrical joint, except that this time we also take into account the constraint equation $\zeta_2 = 0$:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters:
  - six for S_1 : (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - six for S_2 : (ξ_2, η_2, ζ_2) and $(\psi_2, \theta_2, \varphi_2)$.
- Primitive constraint equations: five equations [A2.17].
- Retained parameters:
  - six for S_1 : (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - one for S_2: the spin φ_2.
- No complementary constraint equation.

The VP of the constraint inter-efforts is always given by [5.14]:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{R}_{1\to 2}.\vec{V}^*_{12}(O_2) + \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{\Omega}^*_{12}$$

This time, the same calculation as in the analysis of the cylindrical joint gives $\vec{V}^*_{12}(O_2) = \vec{0}$. On the other hand, the virtual angular velocity $\vec{\Omega}^*_{12}$ is the same as in [A2.14]:

$$\vec{\Omega}^*_{12} = (\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\vec{z}_1$$

where $\dot{\varphi}^*_1, \dot{\varphi}^*_2$ are arbitrary. Consequently, the VP of the constraint inter-efforts becomes

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = (\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\,\vec{\mathcal{M}}_{1\to 2}(O_2).\vec{z}_1 \qquad \blacksquare$$

A2.5. Prismatic or sliding joint

Consider two rigid bodies S_1 and S_2, the first containing a prismatic cavity with axis $O_1\vec{z}_1$ and the second containing a prism with axis $O_2\vec{z}_2$, whose dimensions are slightly smaller than those of the cavity. In Figure A2.5(a), the prismatic cavity and the prism have rectangular cross-sections. The rigid bodies are said to be connected by a prismatic joint when the prism is inserted into the cavity and slides along it, such that their axes coincide throughout the motion (Figure A2.5(b)).

The prismatic joint between two rigid bodies S_1 and S_2 is, therefore, a cylindrical joint with the relative rotation of the two solids around their common axis $O_1\vec{z}_1$ being prevented.

The prismatic joint is, thus, expressed through the relationships [A2.11] to which we add the additional restriction $\varphi_2 = 0$:

$$\begin{cases} O_2 \in \text{ axis } O_1\vec{z}_1 & \Leftrightarrow\ \xi_2 = \eta_2 = 0 \\ \vec{z}_2 = \vec{z}_1 (\Rightarrow \vec{n}_1 = \vec{n}_2) & \Leftrightarrow\ \psi_2 = \psi_1, \theta_2 = \theta_1 \\ \varphi_1 = \varphi_2 & \end{cases} \qquad \text{[A2.19]}$$

which gives five scalar constraint equations. The rigid body S_1 is able to move freely while the only motion possible for S_2 relative to S_1 is the translation along the common axis $O_1\vec{z}_1$.

Theorem. The prismatic joint is *perfect* if and only if

$$\boxed{\vec{R}_{1\to 2} \perp \overrightarrow{z_1}} \qquad \text{[A2.20]}$$

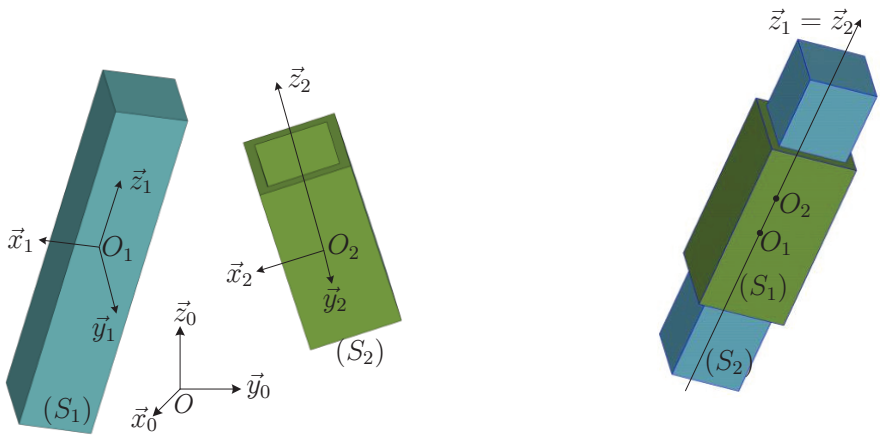

Figure A2.5. *Prismatic joint*

PROOF. We choose the same parameterization as in the analysis of the cylindrical joint, except that this time we also take into account the constraint equation $\varphi_1 = \varphi_2$:

INDEPENDENT PARAMETERIZATION.

- Primitive parameters:
  - six for S_1 : (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - six for S_2 : (ξ_2, η_2, ζ_2) and $(\psi_2, \theta_2, \varphi_2)$.
- Primitive constraint equations: five equations [A2.17].
- Retained parameters:
  - six for S_1: (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - one for S_2: the coordinate ζ_2.
- No complementary constraint equation.

The VP of the constraint inter-efforts is always given by [5.14]:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{R}_{1\to 2}.\vec{V}^*_{12}(O_2) + \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{\Omega}^*_{12}$$

The same calculation as used for the cylindrical joint gives $\vec{V}^*_{12}(O_2) = \dot{\zeta}^*_2\vec{z}_1$ (see [A2.13]). On the other hand, as the virtual angular velocity $\vec{\Omega}^*_{12}$ is

$$\vec{\Omega}_{12} = \vec{\Omega}_{02} - \vec{\Omega}_{01} = (\dot{\varphi}_2 - \dot{\varphi}_1)\vec{z}_1 = \vec{0}$$

it results that

$$\vec{\Omega}^*_{12} = \vec{0}$$

Consequently, the VP of the constraint inter-efforts becomes

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \dot{\zeta}^*_2\ \vec{R}_{1\to 2}.\vec{z}_1$$

where $\dot{\zeta}^*_2$ is arbitrary. ■

A2.6. Helical joint (or screw joint)

The helical (or screw) joint between two rigid bodies S_1 and S_2 is a cylindrical joint with an addition linear relationship between the relative translation of the two rigid bodies and the relative rotation around their common axis $O_1\vec{z}_1$:

$$\zeta_2 = p\,(\varphi_2 - \varphi_1)$$

where p is a given constant.

The helical joint is, thus, expressed by relationships [A2.11] to which we add the aforementioned additional constraint:

$$\begin{cases} O_2 \in \text{ axis } O_1\vec{z}_1 & \Leftrightarrow\ \xi_2 = \eta_2 = 0 \\ \vec{z}_2 = \vec{z}_1 (\Rightarrow \vec{n}_1 = \vec{n}_2) & \Leftrightarrow\ \psi_2 = \psi_1, \theta_2 = \theta_1 \\ \zeta_2 = p\,(\varphi_2 - \varphi_1) \end{cases} \qquad \text{[A2.21]}$$

that is, five scalar constraint equations. The rigid body S_1 is able to move freely, while the rigid body S_2 can translate and rotate relative to S_1, the translation being proportional to the relative rotation.

An example for the device with a helical joint is depicted in Figure A2.6, where S_1 is the bolt and S_2 the screw.

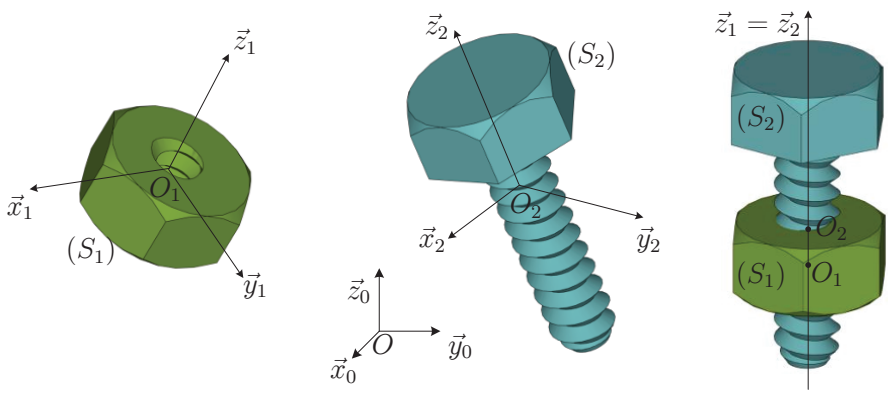

Figure A2.6. *Helical joint*

Let p be a particle of S_2 whose position in R_0 is P, not located on the common axis $O_1\vec{z}_1$. During the motion of the rigid bodies, the point P describes *a circular helix* whose axis is $O_1\vec{z}_1$ attached to S_1 (more precisely, the position $P^{(1)} = Q_{10}P$ of p in R_1 describes a circular helix in the affine 3D space $\mathcal{E}$). The constant $2\pi p$ in $[\text{A2.21}]_3$ is called *the pitch* of the helix or the screw.

Theorem. The helical joint is *perfect* if and only if

$$\boxed{p\vec{R}_{1\to 2}.\vec{z}_1 + \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{z}_1 = 0} \qquad \text{[A2.22]}$$

PROOF. We choose the same parameterization as in the analysis of the cylindrical joint, except that this time we also take into account the constraint equation $[\text{A2.21}]_3$, which is classified as a complementary equation:

PARAMETERIZATION.

- Primitive parameters:
  - six for S_1 : (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - six for S_2 : (ξ_2, η_2, ζ_2) and $(\psi_2, \theta_2, \varphi_2)$.
- Primitive constraint equations: four equations $[\text{A2.21}]_{1-2}$.
- Retained parameters:
  - six for S_1: (x_1, y_1, z_1) and $(\psi_1, \theta_1, \varphi_1)$,
  - two for S_2: the coordinate ζ_2 of the point O_2 on the axis $O_1\vec{z}_1$ and the spin φ_2.
- Complementary constraint equation: relationship $[\text{A2.21}]_3$: $\zeta_2 = p\,(\varphi_2 - \varphi_1)$.

The VP of the constraint inter-efforts is given by [5.14]:

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = \vec{R}_{1\to 2}.\vec{V}^*_{12}(O_2) + \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{\Omega}^*_{12}$$

The same calculation as in the analysis of the cylindrical joint gives [A2.13] and [A2.14]:

$$\vec{V}^*_{12}(O_2) = \dot{\zeta}^*_2\vec{z}_1 \qquad \text{and} \qquad \vec{\Omega}^*_{12} = (\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\vec{z}_1$$

In order for the VVF to be compatible, the parameters $\dot{q}^*_i$ must satisfy $\dot{\zeta}^*_2 = p\,(\dot{\varphi}^*_2 - \dot{\varphi}^*_1)$. Consequently:

$$\vec{V}^*_{12}(O_2) = p\,(\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\vec{z}_1$$

Finally, the VP of the constraint inter-efforts becomes

$$\mathscr{P}^*(\mathcal{F}_{1\leftrightarrow 2}) = (\dot{\varphi}^*_2 - \dot{\varphi}^*_1)\left(p\vec{R}_{1\to 2}.\vec{z}_1 + \vec{\mathcal{M}}_{1\to 2}(O_2).\vec{z}_1\right)$$

where $\dot{\varphi}^*_1, \dot{\varphi}^*_2$ are arbitrary. ■

Bibliography

[1] P. Brousse. *Mécanique analytique*, Vuibert Université, Paris, France, 1981.

[2] P. Germain. *Mécanique, Vols. I and II*, Ellipses, Paris, France, 1986.

[3] M. Mantion. *Problèmes de mécanique analytique – Questions de stabilité*, Vuibert Université, Paris, France, 1981.

[4] P. Rougée. *Mécanique générale*, Vuibert Université, Paris, France, 1982.

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